8.3 Trig Integrals

We'll use substitution to find:

$$\int \sin^n\theta \cos\theta \, d\theta$$

or $\int \cos^n\theta \sin\theta \, d\theta$

Fact $\boxed{\sin^2\theta + \cos^2\theta = 1}$

Ex: $\int \sin^4\theta \underbrace{\cos\theta \, d\theta}_{du}$

$\boxed{\begin{array}{l} u = \sin\theta \\ du = \cos\theta \, d\theta \end{array}}$

$= \int u^4 \, du$

$= \frac{u^5}{5} + C$

$= \frac{\sin^5\theta}{5} + C$

Ex: $\int \sin^4\theta \cos^3\theta \, d\theta$

$= \int \sin^4\theta \, (1-\sin^2\theta) \underbrace{\cos\theta \, d\theta}_{du}$

$\boxed{\begin{array}{l} u = \sin\theta \\ du = \cos\theta \, d\theta \end{array}}$

$= \int u^4 (1-u^2) \, du$

$= \int (u^4 - u^6)\,du$

$= \frac{u^5}{5} - \frac{u^7}{7} + C$

$= \frac{\sin^5\theta}{5} - \frac{\sin^7\theta}{7} + C$

Ex: $\int \sin^5\theta\, d\theta$

$= \int \sin^4\theta \sin\theta\, d\theta$

$u = \cos\theta$
$du = -\sin\theta\, d\theta$
$-du = \sin\theta\, d\theta$

$= \int (1-\cos^2\theta)^2 \sin\theta\, d\theta$

$= -\int (1-u^2)^2\, du$

$= -\int (1 - 2u^2 + u^4)\, du$

$= -\left[u - \frac{2u^3}{3} + \frac{u^5}{5}\right] + C$

$= -\left[\cos\theta - \frac{2}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta\right] + C$

To integrate even powers:

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Ex: $\int \cos^2\theta \, d\theta$

$$= \int \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= \frac{1}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

Ex: $\int \sin^4 3x \, dx$

$$= \int (\sin^2 3x)^2 \, dx$$

$$= \int \left(\frac{1-\cos 6x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int (1 - 2\cos 6x + \cos^2 6x)\, dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 6x + \frac{1}{2} + \frac{\cos 12x}{2}\right) dx$$

$$= \frac{1}{4}\left[\frac{3x}{2} - \frac{\sin 6x}{3} + \frac{\sin 12x}{24}\right] + C$$

To evaluate $\int \sec^m\theta \tan^n\theta \, d\theta$:

$$1 + \tan^2\theta = \sec^2\theta$$

Sub $u = \tan\theta$, $du = \sec^2\theta \, d\theta$

OR $u = \sec\theta$, $du = \sec\theta\tan\theta \, d\theta$

<u>Ex</u>: $\int \tan^3\theta \sec^3\theta \, d\theta$

$$= \int \tan^2\theta \sec^2\theta \underbrace{\sec\theta\tan\theta \, d\theta}_{du}$$

$$\boxed{\begin{array}{l} u = \sec\theta \\ du = \sec\theta\tan\theta \, d\theta \end{array}}$$

$$= \int (\sec^2\theta - 1)\sec^2\theta \sec\theta\tan\theta \, d\theta$$

$$= \int (u^2 - 1)u^2 \, du$$

$$= \int (u^4 - u^2) \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\sec^5\theta}{5} - \frac{\sec^3\theta}{3} + C$$

Ex: $\int \tan^4\theta \, d\theta$

$= \int \tan^2\theta \, (\sec^2\theta - 1) \, d\theta$

$= \int \tan^2\theta \sec^2\theta \, d\theta - \int \tan^2\theta \, d\theta$

$= \int \tan^2\theta \sec^2\theta \, d\theta - \int (\sec^2\theta - 1) \, d\theta$

$= \underbrace{\int \tan^2\theta \sec^2\theta \, d\theta - \int \sec^2\theta \, d\theta}_{\substack{u = \tan\theta \\ du = \sec^2\theta \, d\theta}} + \int d\theta$

$= \int u^2 \, du - \int du + \int d\theta$

$= \dfrac{u^3}{3} - u + \theta + C$

$= \dfrac{\tan^3\theta}{3} - \tan\theta + \theta + C$

To evaluate $\int \csc^m\theta \cot^n\theta \, d\theta$:

$1 + \cot^2\theta = \csc^2\theta$

Sub $u = \csc\theta$, $du = -\csc\theta \cot\theta \, d\theta$

OR $u = \cot\theta$, $du = -\csc^2\theta \, d\theta$

Ex: $\int \cot^2 3x \csc^4 3x \, dx$

$$= \int \cot^2 3x \,(1+\cot^2 3x)\, \underbrace{\csc^2 3x \, dx}$$

$$\boxed{\begin{aligned} u &= \cot 3x \\ du &= -3\csc^2 3x \, dx \\ -\frac{du}{3} &= \csc^2 3x \, dx \end{aligned}}$$

$$= -\frac{1}{3}\int u^2(1+u^2)\, du$$

$$= -\frac{1}{3}\int (u^2+u^4)\, du$$

$$= -\frac{1}{3}\left[\frac{u^3}{3}+\frac{u^5}{5}\right]+C$$

$$= -\frac{1}{3}\left[\frac{\cot^3 3x}{3}+\frac{\cot^5 3x}{5}\right]+C$$

Three Other Techniques

1) Integration by Parts
2) Convert to $\sin\theta$ and $\cos\theta$
3) Identities

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$

Ex: $\int \sec^3\theta \, d\theta$

$= \int \underbrace{\sec\theta}_{u} \underbrace{\sec^2\theta \, d\theta}_{dv}$

$$\boxed{\begin{array}{ll} u = \sec\theta & dv = \sec^2\theta \, d\theta \\ du = \sec\theta\tan\theta \, d\theta & v = \tan\theta \end{array}}$$

$= uv - \int v \, du$

$= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta \, d\theta$

$= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1) \, d\theta$

$= \sec\theta\tan\theta - \int \sec^3\theta \, d\theta + \int \sec\theta \, d\theta$

$= \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| - \int \sec^3\theta \, d\theta$

Let $I = \int \sec^3\theta \, d\theta$

$I = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| - I$

$2I = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C_1$

$I = \frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right] + C$

Ex: $\int \sec\theta \cot^2\theta \, d\theta$

$$= \int \frac{1}{\cos\theta} \frac{\cos^2\theta}{\sin^2\theta} \, d\theta$$

$$= \int \frac{\cos\theta}{\sin^2\theta} \, d\theta$$

$$\boxed{\begin{aligned} u &= \sin\theta \\ du &= \cos\theta \, d\theta \end{aligned}}$$

$$= \int u^{-2} \, du$$

$$= -u^{-1} + C$$

$$= -\frac{1}{\sin\theta} + C$$

$$= -\csc\theta + C$$

Ex: $\int \sin 3x \cos 7x \, dx$

$$= \frac{1}{2} \int [\sin(-4x) + \sin 10x] \, dx$$

$$= \frac{1}{2} \int [-\sin 4x + \sin 10x] \, dx$$

$$= \frac{1}{2} \left[\frac{\cos 4x}{4} - \frac{\cos 10x}{10} \right] + C$$