8.2 Integration by Parts

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Comes from integrating: $uv' + vu' = [uv]'$

Ex: $\int x e^{2x}\,dx$

$$\boxed{\begin{array}{ll} u = x & dv = e^{2x}\,dx \\ du = dx & v = \frac{1}{2}e^{2x} \end{array}}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int x e^{2x}\,dx = \frac{1}{2}x e^{2x} - \int \frac{1}{2}e^{2x}\,dx$$

$$= \frac{1}{2}x e^{2x} - \frac{1}{4}e^{2x} + C$$

Ex: $\int x^3 \ln x\,dx$

$$\boxed{\begin{array}{ll} u = \ln x & dv = x^3\,dx \\ du = \dfrac{dx}{x} & v = \dfrac{x^4}{4} \end{array}}$$

$$\int x^3 \ln x\,dx = \frac{x^4}{4}\ln x - \int \frac{x^3}{4}\,dx$$

$$= \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$$

$\int \arctan x \, dx$

$$\boxed{\begin{array}{ll} u = \arctan x & dv = dx \\ du = \dfrac{1}{1+x^2}\,dx & v = x \end{array}}$$

$$\int \arctan x \, dx = x\arctan x - \int \frac{x}{1+x^2}\,dx$$

$$\boxed{\begin{array}{l} w = 1+x^2 \\ dw = 2x\,dx \\ \dfrac{dw}{2} = x\,dx \end{array}}$$

$$= x\arctan x - \frac{1}{2}\int \frac{dw}{w}$$

$$= x\arctan x - \frac{1}{2}\ln|w| + C$$

$$= x\arctan x - \frac{1}{2}\ln|1+x^2| + C$$

Ex: $\int x\sqrt{1+x}\,dx$

$$\boxed{\begin{array}{ll} u = x & dv = (1+x)^{1/2}\,dx \\ du = dx & v = \frac{2}{3}(1+x)^{3/2} \end{array}}$$

$$\int x\sqrt{1+x}\,dx = \frac{2}{3}x(1+x)^{3/2} - \int \frac{2}{3}(1+x)^{3/2}\,dx$$

$$= \frac{2}{3}x(1+x)^{3/2} - \frac{2}{3}\cdot\frac{2}{5}(1+x)^{5/2} + C$$

$$= \frac{2}{3}x(1+x)^{3/2} - \frac{4}{15}(1+x)^{5/2} + C$$

<u>Ex</u>: $\int e^{2x}\cos x\,dx$

$$\boxed{\begin{array}{ll} u = e^{2x} & dv = \cos x\,dx \\ du = 2e^{2x}\,dx & v = \sin x \end{array}}$$

$$\int e^{2x}\cos x\,dx = e^{2x}\sin x - 2\underbrace{\int e^{2x}\sin x\,dx}_{\text{Integration by Parts again}} \quad (1)$$

$\int e^{2x}\sin x\,dx$

$$\boxed{\begin{array}{ll} u = e^{2x} & dv = \sin x\,dx \\ du = 2e^{2x}\,dx & v = -\cos x \end{array}}$$

$$\int e^{2x}\sin x\,dx = -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx \quad (2)$$

(2) → (1): $\int e^{2x}\cos x\,dx = e^{2x}\sin x - 2\left[-e^{2x}\cos x + 2\int e^{2x}\cos x\,dx\right]$

<u>Let $I = \int e^{2x}\cos x\,dx$</u>

$$I = e^{2x}\sin x + 2e^{2x}\cos x - 4I$$

$$5I = e^{2x}\sin x + 2e^{2x}\cos x \underline{+ C}$$

$$I = \frac{e^{2x}}{5}(\sin x + 2\cos x) + C$$

Tabular Method

Can be useful when integral contains x^n.

Ex: $\int x^2 \cos 3x \, dx$

	D	I
+	x^2	$\cos 3x$
−	$2x$	$\frac{1}{3}\sin 3x$
+	2	$-\frac{1}{9}\cos 3x$
		$-\frac{1}{27}\sin 3x$

$$\int x^2 \cos 3x \, dx = \frac{x^2}{3}\sin 3x + \frac{2x}{9}\cos 3x - \frac{2}{27}\sin 3x \underline{+C}$$

Ex: $\int (x^3+2x)e^{2x}\,dx$

	D	I
$\oplus$	(x^3+2x)	e^{2x}
$\ominus$	$(3x^2+2)$	$e^{2x}/2$
$\oplus$	$6x$	$e^{2x}/4$
$\ominus$	6	$e^{2x}/8$
		$e^{2x}/16$

$$\int (x^3+2x)e^{2x}\,dx = \frac{1}{2}(x^3+2x)e^{2x} - \frac{1}{4}(3x^2+2)e^{2x}$$

$$+ \frac{6x}{8}e^{2x} - \frac{6}{16}e^{2x} \underline{+C}$$

$$= \left[\frac{x^3}{2} + x - \frac{3x^2}{4} - \frac{1}{2} + \frac{3x}{4} - \frac{3}{8}\right]e^{2x} + C$$

$$= \left[\frac{x^3}{2} - \frac{3x^2}{4} + \frac{7x}{4} - \frac{7}{8}\right]e^{2x} + C$$

or $\frac{e^{2x}}{8}\left[4x^3 - 6x^2 + 14x - 7\right] + C$

Ex: $\int 2x^3 \cos x^2 \, dx$

$$\boxed{\begin{aligned} u &= x^2 \\ du &= 2x \, dx \end{aligned}}$$

$= \int x^2 (2x) \cos x^2 \, dx$

$= \int u \cos u \, du$

	D	I
$+$	u	$\cos u$
$-$	1	$\sin u$
		$-\cos u$

$= u \sin u + \cos u + C$

$= x^2 \sin x^2 + \cos x^2 + C$