

5.8 Inverse Trig Functions: Integration

Let $a > 0$.

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + C$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C$$

Ex: Find $\int \frac{1}{\sqrt{16 - x^2}} dx$
 $= \arcsin \frac{x}{4} + C$

Ex: Find $\int \frac{1}{9 + 4x^2} dx$

$$= \frac{1}{2} \int \frac{1}{3^2 + u^2} du$$
$$= \frac{1}{2} \left(\frac{1}{3} \arctan \frac{u}{3} \right) + C$$
$$= \frac{1}{6} \arctan \frac{2x}{3} + C$$

$$u = 2x$$
$$du = 2 dx$$
$$\frac{du}{2} = dx$$

Ex: Evaluate $\int_0^{\pi/2} \frac{\cos \theta}{1 + \sin^2 \theta} d\theta$

$$u = \sin \theta$$
$$du = \cos \theta d\theta$$
$$\theta = 0 \Rightarrow u = 0$$
$$\theta = \frac{\pi}{2} \Rightarrow u = 1$$

$$= \int_0^1 \frac{du}{1+u^2}$$
$$= [\arctan u]_0^1$$
$$= \frac{\pi}{4} - 0$$
$$= \frac{\pi}{4}$$

Ex: Complete the square:

$$x^2 + 10x + 34$$

$$\begin{cases} \frac{10}{2} = 5 \\ 5^2 = 25 \end{cases}$$

$$= x^2 + 10x + 25 + 34 - 25$$
$$= (x+5)^2 + 9$$

Ex: Find $\int \frac{1}{x^2+6x+13} dx$

$$\begin{aligned} & x^2+6x+13 \\ &= x^2+6x+9+4 \\ &= (x+3)^2+2^2 \end{aligned}$$

$$= \int \frac{dx}{(x+3)^2+2^2}$$

$$\begin{aligned} u &= x+3 \\ du &= dx \end{aligned}$$

$$= \int \frac{du}{u^2+2^2}$$

$$= \frac{1}{2} \arctan \frac{u}{2} + C$$

$$= \frac{1}{2} \arctan \frac{x+3}{2} + C$$

Ex: Find $\int \frac{dx}{\sqrt{8x-x^2}}$

$$\begin{aligned} & 8x-x^2 \\ &= -(x^2-8x) \\ &= -(x^2-8x+16)+16 \\ &= 4^2-(x-4)^2 \end{aligned}$$

$$= \int \frac{dx}{\sqrt{4^2-(x-4)^2}}$$

$$\begin{aligned}
 &= \int \frac{du}{\sqrt{4^2 - u^2}} \\
 &= \arcsin \frac{u}{4} + C \\
 &= \arcsin \frac{x-4}{4} + C
 \end{aligned}$$

$$\begin{aligned}
 u &= x-4 \\
 du &= dx
 \end{aligned}$$

Ex: Find $\int \frac{2x+8}{x^2+2x+10} dx$

$$= \int \frac{2x+8}{(x+1)^2+3^2} dx$$

$$\begin{aligned}
 x^2+2x+10 \\
 &= x^2+2x+1+9 \\
 &= (x+1)^2+3^2
 \end{aligned}$$

$$= \int \frac{2u+6}{u^2+3^2} du$$

$$= \int \left[\frac{2u}{u^2+3^2} + \frac{6}{u^2+3^2} \right] du$$

$$= \ln|u^2+9| + \frac{6}{3} \arctan \frac{u}{3} + C$$

$$= \ln|(x+1)^2+9| + 2 \arctan \frac{x+1}{3} + C$$

$$\begin{aligned}
 u &= x+1 \\
 du &= dx \\
 2x+8 &=? \\
 2x+8 &= 2(x+1)+6 \\
 &= 2u+6
 \end{aligned}$$