5.6 Indeterminate Forms

Indeterminate forms:

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0(\infty)$, $\infty - \infty$, 0^0, ∞^0, 1^∞

The limit may or may not exist.

L'Hôpital's Rule

Suppose $\lim_{x \to a} \frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ or $\frac{-\infty}{\infty}$.

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ as long as the second limit exists (or is $\pm\infty$).

Note: $x \to a$ also includes

$x \to a^-$, $x \to a^+$, $x \to \infty$, $x \to -\infty$

Ex: $\displaystyle\lim_{x\to\infty} \frac{x^2+1}{3x^2+5x+4}$

The form is $\frac{\infty}{\infty}$ ✓

$$\lim_{x\to\infty} \frac{x^2+1}{3x^2+5x+4} \overset{H}{=} \lim_{x\to\infty} \frac{2x}{6x+5}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{2}{6}$$

$$= \frac{1}{3}$$

The form must be $\frac{0}{0}$ or $\frac{\infty}{\infty}$ or $\frac{-\infty}{\infty}$ each time the rule is applied.

Ex: $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$

The form is $\frac{0}{0}$ ✓

$$\lim_{x\to 0} \frac{\sin x}{x} \overset{H}{=} \lim_{x\to 0} \frac{\cos x}{1}$$

$$= 1$$

$\underline{Ex}$: $\displaystyle\lim_{x\to\infty} \frac{3x^2+7x+1}{e^{2x}}$

The form is $\frac{\infty}{\infty}$ ✓

$$\lim_{x\to\infty} \frac{3x^2+7x+1}{e^{2x}} \stackrel{H}{=} \lim_{x\to\infty} \frac{6x+7}{2e^{2x}}$$

$$\stackrel{H}{=} \lim_{x\to\infty} \frac{6}{4e^{2x}}$$

$$= 0$$

$\underline{Ex}$: $\displaystyle\lim_{x\to 2} \frac{1-\cos(x^3-8)}{(x-2)^2}$

The form is $\frac{0}{0}$ ✓

$$\lim_{x\to 2} \frac{1-\cos(x^3-8)}{(x-2)^2} \stackrel{H}{=} \lim_{x\to 2} \frac{3x^2\sin(x^3-8)}{2(x-2)}$$

$$\stackrel{H}{=} \lim_{x\to 2} \frac{9x^4\cos(x^3-8)+6x\sin(x^3-8)}{2}$$

$$= \frac{144}{2}$$

$$= 72$$

FACT

$$\boxed{\lim_{x\to\infty} \frac{P_n(x)}{e^{ax}} = 0 \quad \text{where: } P_n(x) \text{ is a polynomial of degree } n \text{ and } a>0.}$$

Ex: $\lim\limits_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

The form is $\infty(0)$. Rewrite.

$$\lim_{x\to\infty} x\sin\left(\tfrac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)}$$

$$= \lim_{x\to\infty} \cos\left(\tfrac{1}{x}\right)$$

$$= 1$$

Ex: $\displaystyle\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$

The form is $\infty-\infty$. Rewrite.

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1^+}\frac{x-1-\ln x}{(x-1)\ln x}$$

$$\overset{H}{=}\lim_{x\to 1^+}\frac{\left(1-\frac{1}{x}\right)}{(x-1)\frac{1}{x}+\ln x}$$

$$\overset{H}{=}\lim_{x\to 1^+}\frac{\left(\frac{1}{x^2}\right)}{(x-1)\left(\frac{-1}{x^2}\right)+\frac{1}{x}+\frac{1}{x}}$$

$$=\frac{1}{2}$$

We use logarithms to deal with the forms 0^0, ∞^0, and 1^∞.

Ex: $\lim_{x \to 0^+} x^x$

Let $L = \lim_{x \to 0^+} x^x$

$$\ln L = \ln \left(\lim_{x \to 0^+} x^x \right)$$

$$= \lim_{x \to 0^+} \ln x^x$$

$$= \lim_{x \to 0^+} x \ln x$$

$$= \lim_{x \to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} \leftarrow \text{form is } \frac{-\infty}{\infty} \checkmark$$

$$\overset{H}{=} \lim_{x \to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)}$$

$$= \lim_{x \to 0^+} -x$$

$$= 0$$

$$\ln L = 0$$

$$\Rightarrow L = e^0 = 1$$

Ex: $\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x$

Let $L = \lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x$

$$\ln L = \lim_{x \to \infty} \ln\left(1+\frac{1}{x}\right)^x$$

$$= \lim_{x \to \infty} x \ln\left(1+\frac{1}{x}\right)$$

$$= \lim_{x \to \infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} \leftarrow \text{Form is } \frac{0}{0} \checkmark$$

$$\overset{H}{=} \lim_{x \to \infty} \frac{\left(\frac{1}{1+\frac{1}{x}}\right)\left(\frac{-1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)}$$

$$= \lim_{x \to \infty} \frac{1}{1+\frac{1}{x}}$$

$$= 1$$

$$\ln L = 1$$

$$\Rightarrow L = e^1 = e$$

Ex: $\lim_{x \to \frac{\pi}{2}^-} \tan x^{\cos x}$

Let $L = \lim_{x \to \frac{\pi}{2}^-} (\tan x)^{\cos x}$

$$\ln L = \lim_{x \to \frac{\pi}{2}^-} \ln(\tan x^{\cos x})$$

$$= \lim_{x \to \frac{\pi}{2}^-} (\cos x)(\ln \tan x)$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\ln \tan x}{\sec x} \Leftarrow \text{Form is } \frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{x \to \frac{\pi}{2}^-} \frac{\frac{1}{\tan x}(\sec^2 x)}{\sec x \tan x}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\sec x}{\tan^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{1}{\cos x} \frac{\cos^2 x}{\sin^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\cos x}{\sin^2 x}$$

$$= 0$$

$$\ln L = 0$$

$$\Rightarrow L = e^0 = 1$$