5.2 and 5.4 Exponentials and Logs: Integration

$$\boxed{\int \frac{1}{u}\,du = \ln|u| + C}$$

Ex: $\int \frac{x}{x^2+4}\,dx$

$$\boxed{\begin{aligned} u &= x^2+4 \\ du &= 2x\,dx \\ \frac{du}{2} &= x\,dx \end{aligned}}$$

$= \frac{1}{2}\int \frac{du}{u}$

$= \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|x^2+4| + C$

Ex: $\int \frac{1}{3x+7}\,dx$

$$\boxed{\begin{aligned} u &= 3x+7 \\ du &= 3\,dx \\ \frac{du}{3} &= dx \end{aligned}}$$

$= \frac{1}{3}\int \frac{du}{u}$

$= \frac{1}{3}\ln|u| + C$

$= \frac{1}{3}\ln|3x+7| + C$

$$\boxed{\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C \quad (a \neq 0)}$$

Long Division $f(x) = \dfrac{3x^3 - 5x^2 + 10x - 3}{3x+1} = ?$

$$
\begin{array}{r}
x^2 - 2x + 4 \\
3x+1 \overline{\smash{)}\, 3x^3 - 5x^2 + 10x - 3} \\
-(3x^3 + x^2) \\
\hline
-6x^2 + 10x - 3 \\
-(-6x^2 - 2x) \\
\hline
12x - 3 \\
-(12x + 4) \\
\hline
-7
\end{array}
$$

$$f(x) = x^2 - 2x + 4 + \frac{-7}{3x+1}$$

Ex: $\displaystyle\int \frac{3x^3 - 5x^2 + 10x - 3}{3x+1}\,dx$

$$= \int \left(x^2 - 2x + 4 + \frac{-7}{3x+1}\right) dx$$

$$= \frac{x^3}{3} - x^2 + 4x - \frac{7}{3}\ln|3x+1| + C$$

Ex: $\int \cot \frac{\theta}{4}\, d\theta$

$= \int \frac{\cos \frac{\theta}{4}}{\sin \frac{\theta}{4}}\, d\theta$

$u = \sin \frac{\theta}{4}$

$du = \frac{1}{4}\cos \frac{\theta}{4}\, d\theta$

$4du = \cos \frac{\theta}{4}\, d\theta$

$= 4\int \frac{du}{u}$

$= 4 \ln|u| + C$

$= 4 \ln\left|\sin \frac{\theta}{4}\right| + C$

Ex: $\int \frac{(\ln x)^3}{x}\, dx$

$u = \ln x$

$du = \frac{1}{x}\, dx$

$= \int u^3\, du$

$= \frac{u^4}{4} + C$

$= \frac{1}{4}(\ln x)^4 + C$

Ex: $\int \frac{3x}{(x-2)^2}\,dx$

$$u = x-2$$
$$du = dx$$
$$x = ?$$
$$x = u+2$$

$$= 3\int \frac{u+2}{u^2}\,du$$

$$= 3\int\left(\frac{1}{u}+\frac{2}{u^2}\right)du$$

$$= 3\left[\ln|u| - 2u^{-1}\right] + C$$

$$= 3\left[\ln|x-2| - 2(x-2)^{-1}\right] + C$$

Ex: $\int \frac{\sqrt{x}}{\sqrt{x}-1}\,dx$

$$u = \sqrt{x}-1$$
$$du = \tfrac{1}{2}x^{-1/2}\,dx$$
$$\sqrt{x}\,dx = ?$$
$$2\,du = \frac{dx}{\sqrt{x}}$$
$$2x\,du = \sqrt{x}\,dx$$
$$2(u+1)^2\,du = \sqrt{x}\,dx$$

$$= 2\int \frac{(u+1)^2\,du}{u}$$

$$= 2\int \frac{u^2+2u+1}{u}\,du$$

$$= 2\int \left(u + 2 + \frac{1}{u}\right) du$$

$$= 2\left[\frac{u^2}{2} + 2u + \ln|u|\right] + C$$

$$= 2\left[\frac{(\sqrt{x}-1)^2}{2} + 2(\sqrt{x}-1) + \ln|\sqrt{x}-1|\right] + C$$

$$\boxed{\int e^u\,du = e^u + C}$$

Ex: $\int e^{8x+4}\,dx$

$$\boxed{\begin{aligned} u &= 8x+4 \\ du &= 8\,dx \\ \frac{du}{8} &= dx \end{aligned}}$$

$$= \frac{1}{8}\int e^u\,du$$

$$= \frac{1}{8}e^u + C$$

$$= \frac{1}{8}e^{8x+4} + C$$

$$\boxed{\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C \quad (a \neq 0)}$$

Ex: $\int \frac{4e^{3x}}{2+5e^{3x}}\,dx$

$$\boxed{\begin{aligned} u &= 2+5e^{3x} \\ du &= 15e^{3x}\,dx \\ \frac{du}{15} &= e^{3x}\,dx \end{aligned}}$$

$$= \frac{4}{15}\int \frac{du}{u}$$

$$= \frac{4}{15}\ln|u| + C$$

$$= \frac{4}{15}\ln|2+5e^{3x}| + C$$

Ex: $\int e^x\sqrt{1+e^x}\,dx$

$$\boxed{\begin{aligned} u &= 1+e^x \\ du &= e^x\,dx \end{aligned}}$$

$$= \int \sqrt{u}\,du$$

$$= \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{3}(1+e^x)^{3/2} + C$$