12.2 Derivatives and Integrals of Vector-Valued Functions

FACT

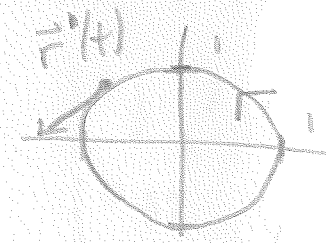
If $\vec{r}(t) = [x(t), y(t), z(t)]$

then $\vec{r}\,'(t) = [x'(t), y'(t), z'(t)]$

Note: $\vec{r}\,'(t)$ is a tangent vector to the curve.

Ex: Find $\vec{r}\,'(t)$ for $\vec{r}(t) = [\cos t, \sin t]$

$\vec{r}\,'(t) = [-\sin t, \cos t]$



6 Properties

Let: c be a constant

$f(t)$ be a function of t

$\vec{r}(t), \vec{s}(t)$ be vector-valued functions

1) $[c\vec{r}(t)]' = c\vec{r}\,'(t)$

2) $[\vec{r}(t) \pm \vec{s}(t)]' = \vec{r}\,'(t) \pm \vec{s}\,'(t)$

3) $[f(t)\vec{r}(t)]' = f(t)\vec{r}\,'(t) + f'(t)\vec{r}(t)$ Product Rule

4) $[\vec{r}(t)\cdot\vec{s}(t)]' = \vec{r}(t)\cdot\vec{s}\,'(t) + \vec{r}\,'(t)\cdot\vec{s}(t)$ Product Rule

5) $[\vec{r}(t)\times\vec{s}(t)]' = \vec{r}(t)\times\vec{s}\,'(t) + \vec{r}\,'(t)\times\vec{s}(t)$ Product Rule

6) $\frac{d}{dt}[\vec{r}(f(t))] = \vec{r}\,'(f(t))\,f'(t)$ Chain Rule

Ex: Find $\frac{d}{dt}\, t^2\langle 9t, t^3\rangle$ two different ways.

$$\frac{d}{dt}\, t^2\langle 9t, t^3\rangle = \frac{d}{dt}\langle 9t^3, t^5\rangle$$
$$= \langle 27t^2, 5t^4\rangle$$

$$\frac{d}{dt}\, t^2\langle 9t, t^3\rangle = t^2\langle 9, 3t^2\rangle + 2t\langle 9t, t^3\rangle$$
$$= \langle 9t^2, 3t^4\rangle + \langle 18t^2, 2t^4\rangle$$
$$= \langle 27t^2, 5t^4\rangle$$

Ex: Let $\vec{r}(t) = [t^2+1, 7t]$

Find $\frac{d}{dt}\vec{r}(2t)$ two different ways.

$$\frac{d}{dt}\vec{r}(2t) = \frac{d}{dt}[4t^2+1, 14t]$$
$$= [8t, 14]$$

$$\vec{r}'(t) = [2t, 7]$$
$$\vec{r}'(2t) = [4t, 7]$$
$$\frac{d}{dt}\vec{r}(2t) = \vec{r}'(2t)\,(2)$$
$$= 2[4t, 7]$$
$$= [8t, 14]$$

Ex: $\vec{r}(t) = [t^2, 7t, t^3]$

Find $\vec{r}'(t) \cdot \vec{r}''(t)$

and $\vec{r}'(t) \times \vec{r}''(t)$

$$\vec{r}'(t) = [2t, 7, 3t^2]$$
$$\vec{r}''(t) = [2, 0, 6t]$$

$$\vec{r}'(t) \cdot \vec{r}''(t) = 2t(2) + 7(0) + 3t^2(6t)$$
$$= 4t + 18t^3$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 7 & 3t^2 \\ 2 & 0 & 6t \end{vmatrix} \qquad [+ - +]$$

$$= \vec{i}[42t] - \vec{j}[12t^2 - 6t^2] + \vec{k}[-14]$$

$$= (42t)\vec{i} + (-6t^2)\vec{j} - 14\vec{k}$$

$$\text{or} \quad [42t, -6t^2, -14]$$

FACT

If $\vec{r}(t) = [x(t), y(t), z(t)]$

then $\int \vec{r}(t)\,dt = [\int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt]$

Ex: Find $\vec{r}(t)$ if

$$\vec{r}'(t) = [3t^2, 2t, 6e^{2t}]$$

and $\vec{r}(0) = [3, 2, 8]$

$$\vec{r}(t) = [t^3 + C_1, t^2 + C_2, 3e^{2t} + C_3]$$

$$\vec{r}(t) = [t^3, t^2, 3e^{2t}] + \vec{C}$$

$t=0$: $\quad [3,2,8] = [0,0,3] + \vec{C}$

$$\vec{C} = [3,2,5]$$

$$\vec{r}(t) = [t^3, t^2, 3e^{2t}] + [3,2,5]$$

or $\quad \vec{r}(t) = [t^3+3, t^2+2, 3e^{2t}+5]$

Ex. Find $\int_1^2 [4t, 7]\,dt$

$= [2t^2, 7t]\Big|_1^2$

$= [8, 14] - [2, 7]$

$= [6, 7]$