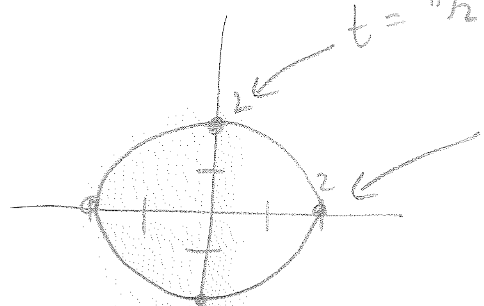
10.3 Parametric Curves and Calculus

Parametric curve example:

$$\begin{cases} x = 2\cos t \\ y = 2\sin t \\ 0 \le t < 2\pi \end{cases}$$



FACT

$$\boxed{\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}, \quad \text{if } \frac{dx}{dt} \neq 0}$$

Comes from the Chain Rule: $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

Ex: Find the slope of the tangent line at $t=1$:

$$\begin{cases} x = 2t^2 + 1 \\ y = t^3 + t^5 \\ -\infty < t < \infty \end{cases}$$

$$\frac{dy}{dt} = 3t^2 + 5t^4$$

$$\frac{dx}{dt} = 4t$$

$$\frac{dy}{dx} = \frac{3t^2 + 5t^4}{4t}$$

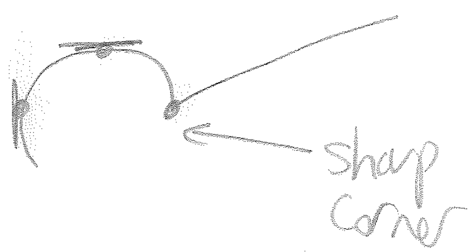
$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{8}{4} = 2$$

FACT

If $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$ there is a horizontal tangent

If $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$ " vertical "

If $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ there <u>may</u> be a sharp corner.



Ex: $\begin{cases} x = 4t + t^4 \\ y = 1 + t^2 \\ -\infty < t < \infty \end{cases}$

Find all points (x, y) where there is a horizontal or vertical tangent.

$$\frac{dx}{dt} = 4 + 4t^3 \qquad \frac{dy}{dt} = 2t$$

$$4 + 4t^3 = 0 \qquad 2t = 0$$

$$t^3 = -1 \qquad t = 0$$

$$t = -1$$

Horizontal Tangent

$\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$

$\Rightarrow t = 0$

$\Rightarrow (x, y) = (0, 1)$

Vertical Tangent

$\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

$\Rightarrow t = -1$

$\Rightarrow (x, y) = (-3, 2)$

$$\boxed{\text{The second derivative} \quad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\left(\frac{dx}{dt}\right)}}$$

Note: Curve is <u>concave up</u> if $\frac{d^2y}{dx^2} > 0$ $\cup$

" <u>concave down</u> if $\frac{d^2y}{dx^2} < 0$ $\cap$

<u>Ex</u>:
$$\begin{cases} x = t^2+1 \\ y = t^6+5 \\ -\infty < t < \infty \end{cases}$$

Find $\frac{d^2y}{dx^2}$

$\frac{dx}{dt} = 2t$ $\quad$ $\frac{dy}{dt} = 6t^5$ $\quad$ $\frac{dy}{dx} = \frac{6t^5}{2t} = 3t^4$

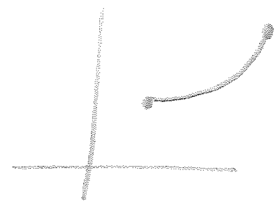
$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{\frac{d}{dt}\left[3t^4\right]}{2t}$$

$$= \frac{12t^3}{2t}$$

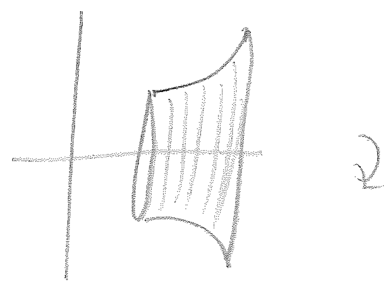
$$= 6t^2$$

Consider $\begin{cases} x = x(t) \\ y = y(t) \\ a \le t \le b \end{cases}$



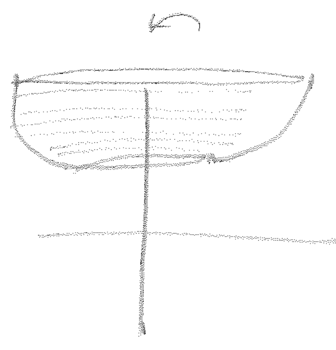
$$\boxed{\text{Arc length} \quad s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Revolve curve about x-axis



(Lateral) Surface area

$$S_x = 2\pi \int_a^b y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$



Revolve curve about y-axis

(Lateral) Surface area

$$S_y = 2\pi \int_a^b x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex: Find the arc length of:

$$\begin{cases} x = t^2 \\ y = t^3 \\ 1 \le t \le 2 \end{cases}$$

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 3t^2$$

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_1^2 \sqrt{(2t)^2 + (3t^2)^2}\, dt$$

$$= \int_1^2 \sqrt{4t^2 + 9t^4}\, dt$$

$$= \int_1^2 \sqrt{t^2(4+9t^2)}\, dt$$

$$= \int_1^2 \sqrt{t^2}\sqrt{4+9t^2}\, dt$$

t (if $t \ge 0$)

$$= \int_1^2 t\sqrt{4+9t^2}\, dt$$

$$\boxed{\begin{aligned} u &= 4+9t^2 \\ du &= 18t\,dt \\ \frac{du}{18} &= t\,dt \\ t=1 &\Rightarrow u=13 \\ t=2 &\Rightarrow u=40 \end{aligned}}$$

$$= \frac{1}{18}\int_{13}^{40} u^{1/2}\, du$$

$$= \frac{1}{18} \cdot \frac{2}{3} u^{3/2} \Big|_{13}^{40}$$

$$= \frac{40\sqrt{40} - 13\sqrt{13}}{27}$$

<u>Ex</u>: Revolve the curve about the x-axis. Find the surface area.

$$\begin{cases} x = 1 + t \\ y = 2\sqrt{t} \\ 0 \le t \le 4 \end{cases}$$

$$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = t^{-1/2}$$

$$S_x = 2\pi \int_a^b y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\pi \int_0^4 2\sqrt{t} \sqrt{1 + t^{-1}}\, dt$$

$$= 4\pi \int_0^4 \sqrt{t+1}\, dt$$

$$\boxed{\begin{array}{l} u = t+1 \\ du = dt \\ t = 0 \Rightarrow u = 1 \\ t = 4 \Rightarrow u = 5 \end{array}}$$

$$= 4\pi \int_1^5 \sqrt{u}\, du$$

$$= 4\pi \, \frac{2}{3} \, u^{3/2} \Big|_1^5$$

$$= \frac{8\pi\,(5\sqrt{5}-1)}{3}$$

Ex: Revolve the curve about the y-axis. Find the surface area.

$$\begin{cases} x = a\sin^3\theta \\ y = a\cos^3\theta \\ 0 \le \theta \le \frac{\pi}{2} \end{cases} \rightarrow \begin{cases} x = a\sin^3 t \\ y = a\cos^3 t \\ 0 \le t \le \frac{\pi}{2} \end{cases}$$

$$\frac{dx}{dt} = 3a\sin^2 t\cos t \qquad \frac{dy}{dt} = -3a\cos^2 t\sin t$$

$$S_y = 2\pi \int_a^b x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\pi \int_0^{\pi/2} a\sin^3 t \sqrt{9a^2\sin^4 t\cos^2 t + 9a^2\cos^4 t\sin^2 t}\, dt$$

$$= 2\pi \int_0^{\pi/2} a\sin^3 t \sqrt{9a^2\sin^2 t\cos^2 t\,\underline{(\sin^2 t + \cos^2 t)}}\, dt$$

$$= 2\pi \int_0^{\pi/2} a\sin^3 t \,|3a\sin t\cos t|\, dt$$

$$= 2\pi \int_0^{\pi/2} a\sin^3 t\,(3a\sin t\cos t)\, dt$$

$$= 2\pi \int_0^{\pi/2} 3a^2 \sin^4 t \cos t \, dt$$

$u = \sin t$

$du = \cos t \, dt$

$t = 0 \Rightarrow u = 0$

$t = \pi/2 \Rightarrow u = 1$

$$= 2\pi \int_0^1 3a^2 u^4 \, du$$

$$= 2\pi (3a^2) \frac{u^5}{5} \Big|_0^1$$

$$= \frac{6a^2\pi}{5}$$