1. [4 marks] Given $\vec{r}(t) = [4\sqrt{t}, \frac{2}{t}, 3t - 5]$. Find:

a) $\vec{r}\,'(t) \cdot \vec{r}\,''(t)$

$$\vec{r}\,'(t) = [2t^{-1/2}, -2t^{-2}, 3]$$

$$\vec{r}\,''(t) = [-t^{-3/2}, 4t^{-3}, 0]$$

$$\vec{r}\,'(t) \cdot \vec{r}\,''(t) = -2t^{-2} - 8t^{-5}$$

b) $\vec{r}\,'(t) \times \vec{r}\,''(t)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t^{-1/2} & -2t^{-2} & 3 \\ -t^{-3/2} & 4t^{-3} & 0 \end{vmatrix}$$

$$= \vec{i}\,(-12t^{-3}) - \vec{j}\,(3t^{-3/2}) + \vec{k}\,(8t^{-7/2} - 2t^{-7/2})$$

$$= [-12t^{-3}, -3t^{-3/2}, 6t^{-7/2}]$$

2. [3 marks] Find $\frac{d^2y}{dx^2}$ given:

$$\begin{cases} x = 4t^3 + 1 \\ y = 3t^2 + 2t + 6 \end{cases}$$

$$\frac{dx}{dt} = 12t^2$$

$$\frac{dy}{dt} = 6t + 2$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{6t+2}{12t^2}$$

$$= \frac{1}{2t} + \frac{1}{6t^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}$$

$$= \frac{\left(-\frac{1}{2}t^{-2} - \frac{1}{3}t^{-3}\right)}{\left(12t^2\right)}$$

$$= \frac{-1}{24}t^{-4} - \frac{1}{36}t^{-5}$$

3. [3 marks] Rewrite as a power series centred at $c = -1$:

$f(x) = \frac{2}{5x+4}$

$$= \frac{2}{5(x+1)-1}$$

$$= \frac{-2}{1-5(x+1)}$$

$$= -2\left[\frac{1}{1-5(x+1)}\right]$$

$$= -2\sum_{n=0}^{\infty}\left[5(x+1)\right]^n$$

4. [5 marks] Use three nonzero terms of an appropriate series to approximate $\int_0^{0.8} x^2\sqrt{1+x^2}\,dx$.

$$(1+x)^{1/2} \approx 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{} x^2$$

$$\approx 1 + \frac{x}{2} - \frac{x^2}{8}$$

$$(1+x^2)^{1/2} \approx 1 + \frac{x^2}{2} - \frac{x^4}{8}$$

$$x^2(1+x^2)^{1/2} \approx x^2 + \frac{x^4}{2} - \frac{x^6}{8}$$

$$\int_0^{0.8} x^2(1+x^2)^{1/2}\,dx \approx \int_0^{0.8}\left(x^2 + \frac{x^4}{2} - \frac{x^6}{8}\right)dx$$

$$\approx \left[\frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{56}\right]_0^{0.8}$$

$$\approx 0.20$$

5. [4 marks] Find $\frac{dy}{dx}$ given $r = 1 - \sin\theta$.

$$x = r\cos\theta$$
$$= (1-\sin\theta)\cos\theta$$

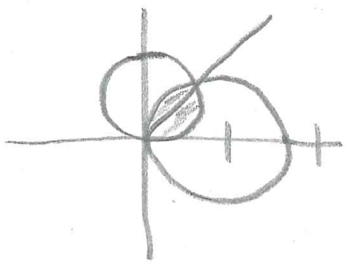
$$\frac{dx}{d\theta} = (1-\sin\theta)(-\sin\theta) + \cos\theta(-\cos\theta)$$
$$= -\sin\theta + \sin^2\theta - \cos^2\theta$$

$$y = r\sin\theta$$
$$= (1-\sin\theta)\sin\theta$$
$$= \sin\theta - \sin^2\theta$$

$$\frac{dy}{d\theta} = \cos\theta - 2\sin\theta\cos\theta$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$
$$= \frac{\cos\theta - 2\sin\theta\cos\theta}{\sin^2\theta - \cos^2\theta - \sin\theta}$$

6. [6 marks] Find the area inside both circles: $r = \sin\theta$ and $r = \sqrt{3}\cos\theta$.



Intersection: $r = r$

$\sin\theta = \sqrt{3}\cos\theta$

$\tan\theta = \sqrt{3}$

$\theta = \frac{\pi}{3}$

$A = A_1 + A_2$

where $A_1 = \frac{1}{2}\int_0^{\pi/3} \sin^2\theta \, d\theta$

$= \frac{1}{2}\int_0^{\pi/3} \left(\frac{1}{2} - \frac{\cos 2\theta}{2}\right) d\theta$

$= \frac{1}{2}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/3}$

$= \frac{1}{2}\left[\frac{\pi}{6} - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)\right]$

$= \frac{\pi}{12} - \frac{\sqrt{3}}{16}$

$A_2 = \frac{1}{2}\int_{\pi/3}^{\pi/2} 3\cos^2\theta \, d\theta$

$= \frac{3}{2}\int_{\pi/3}^{\pi/2} \left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta \rightarrow$

$$= \frac{3}{2}\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{\pi/3}^{\pi/2}$$

$$= \frac{3}{2}\left[\frac{\pi}{4} - \frac{\pi}{6} - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{3}{2}\left[\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right]$$

$$= \frac{\pi}{8} - \frac{3\sqrt{3}}{16}$$

$$A = \frac{\pi}{12} - \frac{\sqrt{3}}{16} + \frac{\pi}{8} - \frac{3\sqrt{3}}{16}$$

$$= \frac{5\pi}{24} - \frac{\sqrt{3}}{4}$$