Math 222 Test Two Total: 25 marks

Name:____________________________

1. [4 marks] Show that $\frac{7n^8+3n^3}{n^2+2}$ is $O(n^6)$.

$$\left|\frac{7n^8+3n^3}{n^2+2}\right| \le \frac{7n^8+3n^3}{n^2+2} \quad \text{for } n>0$$

$$\le \frac{7n^8+3n^8}{n^2+2} \quad \text{for } n>1$$

$$\le \frac{10n^8}{n^2} \quad \text{for } n>1$$

$$\le 10n^6 \quad \text{for } n>1$$

$$\Rightarrow \left|\frac{7n^8+3n^3}{n^2+2}\right| \le 10|n^6| \quad \text{for } n>1$$

$$\Rightarrow \frac{7n^8+3n^3}{n^2+2} \text{ is } O(n^6).$$

2. [4 marks] For each of the pairs of numbers below, find q and r so that $a = qd + r$ and $0 \le r < d$.

a) $a = 107200, d = 845$

$r = 730$

quotient $q = \left\lfloor \frac{107200}{845} \right\rfloor = 126$

$a = 126(845) + 730$

remainder

b) $a = -4512, d = 33$

$r = 9$

quotient $q = \left\lfloor \frac{-4512}{33} \right\rfloor = -137$

$a = -137(33) + 9$

remainder

3. [3 marks] Compute $6^{28} \pmod{11}$ using successive squaring.

$6^2 \equiv 3 \pmod{11}$

$6^4 \equiv 9 \pmod{11}$

$6^8 \equiv 4 \pmod{11}$

$6^{16} \equiv 5 \pmod{11}$

square and reduce mod 11

$$\begin{aligned} 6^{28} &\equiv 6^{16} \cdot 6^{8} \cdot 6^{4} \pmod{11} \\ &\equiv 5 \cdot 4 \cdot 9 \pmod{11} \\ &\equiv 180 \pmod{11} \\ &\equiv 4 \pmod{11} \end{aligned}$$

4. [5 marks] a) Use the Euclidean Algorithm to find gcd(189, 23).

$189 = 8(23) + 5$
$23 = 4(5) + 3$
$5 = 1(3) + 2$
$3 = 1(2) + 1 \Leftarrow$ last nonzero remainder is the gcd
$2 = 2(1) + 0$

$\gcd(189, 23) = 1$

b) Use part a) to find integers x and y so that $189x + 23y = 1$.

$1 = 3 - 1(2)$ Above
$1 = 3 - 1[5 - 1(3)]$ Above
$1 = 2(3) - 5$ Rewrite
$1 = 2[23 - 4(5)] - 5$ Above
$1 = 2(23) - 9(5)$ Rewrite
$1 = 2(23) - 9[189 - 8(23)]$ Above
$1 = 74(23) - 9(189)$ Rewrite

$x = -9 \quad y = 74$

5. [3 marks] Use mathematical induction to show that
$\frac{2}{3}+\frac{2}{9}+\cdots+\frac{2}{3^n}=1-(\frac{1}{3})^n$ for $n\geq 1$.

Basis Step $P(1)$: $\frac{2}{3}=1-\frac{1}{3}$ ✓

Let $k\geqslant 1$.

Induction Step $P(k)$: $\frac{2}{3}+\frac{2}{9}+\cdots+\frac{2}{3^k}=1-\left(\frac{1}{3}\right)^k$

$$\Rightarrow \frac{2}{3}+\frac{2}{9}+\cdots+\frac{2}{3^{k+1}}=\frac{2}{3}+\cdots+\frac{2}{3^k}+\frac{2}{3^{k+1}}$$

$$=1-\left(\frac{1}{3}\right)^k+\frac{2}{3^{k+1}} \quad (*)$$

$$=1-\frac{1}{3^k}\cdot\frac{3}{3}+\frac{2}{3^{k+1}}$$

$$=1-\frac{1}{3^{k+1}}$$

$$=1-\left(\frac{1}{3}\right)^{k+1}$$

This is $P(k+1)$ ✓

6. [6 marks] Find recursive definitions for the following sequences:

a) the sequence $a_1, a_2, a_3, a_4, a_5, \ldots$ given by $5, 7, 9, 11, 13, \ldots$

$\boxed{a_1 = 5 \qquad a_n = a_{n-1} + 2 \quad \text{for } n \geq 2}$

b) the sequence $a_1, a_2, a_3, a_4, a_5, \ldots$ given by $21, 23, 21, 23, 21, \ldots$

$\boxed{a_1 = 21 \qquad a_n = 44 - a_{n-1} \quad \text{for } n \geq 2}$

OR: $\boxed{a_1 = 21 \quad a_2 = 23 \qquad a_n = a_{n-2} \quad \text{for } n \geq 3}$

c) the sequence $a_0, a_1, a_2, a_3, a_4, a_5, a_6, \ldots$ given by $0, 1, 1, 2, 3, 5, 8, \ldots$

$$\boxed{\begin{array}{l} a_0 = 0 \quad a_1 = 1 \\ a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 2 \end{array}}$$