Solutions to Practice Problems 2

(1)
$$
\begin{aligned}
|n^6-17n^3+16n^2| &\le |n^6| + |-17n^3| + |16n^2| && \text{for all } n \\
&\le n^6 + 17n^3 + 16n^2 && \text{for } n>0 \\
&\le n^6 + 17n^6 + 16n^6 && \text{for } n>1 \\
&\le 34n^6 && \text{for } n>1
\end{aligned}
$$

$\Rightarrow$ $|n^6-17n^3+16n^2| \le 34|n^6|$ for $n>1$

$\Rightarrow$ $n^6-17n^3+16n^2$ is $O(n^6)$.

(2)
$$
\begin{aligned}
\left|\frac{5n^3+6}{n+13}\right| &\le \frac{5n^3+6}{n+13} && \text{for } n>0 \\
&\le \frac{5n^3+6n^3}{n+13} && \text{for } n>1 \\
&\le \frac{5n^3+6n^3}{n} && \text{for } n>1 \\
&\le \frac{11n^3}{n} && \text{for } n>1 \\
&\le 11n^2 && \text{for } n>1
\end{aligned}
$$

$\Rightarrow$ $\left|\frac{5n^3+6}{n+13}\right| \le 11|n^2|$ for $n>1$

$\Rightarrow$ $\frac{5n^3+6}{n+13}$ is $O(n^2)$.

③ a) quotient $q = \left\lfloor \frac{137}{29} \right\rfloor = 4$

$137 = 4(29) + \underset{\substack{\uparrow \\ \text{remainder}}}{21}$

$r = 21$

b) $q = \left\lfloor \frac{10824}{36} \right\rfloor = 300$

$10824 = 300(36) + 24$

$r = 24$

c) $q = \left\lfloor \frac{-269}{17} \right\rfloor = -16$ remember: floor means go <u>left</u> on the number line

$-269 = -16(17) + 3$

$r = 3$

④ a) $429075 \cdot 312018 \pmod{11}$

$\equiv 9 \cdot 3 \pmod{11}$

$\equiv 27 \pmod{11}$

$\equiv 5 \pmod{11}$

b) $-9012 \cdot 10400 + 879 \pmod{36}$ $-9012 = -251(36) + 24$

$\equiv 24 \cdot 32 + 15 \pmod{36}$ $-9012 \equiv 24 \pmod{36}$

$\equiv 783 \pmod{36}$

$\equiv 27 \pmod{36}$

(5) Since $x \equiv 67 \pmod{101}$

x could be:

$\underbrace{67, \quad 168}_{+101}, \quad \underbrace{\quad 269}_{+101}, \ldots$ etc.

Continue the list until you find a number $\equiv 5 \pmod 8$

$67, 168, \boxed{269}$

$269 \equiv 5 \pmod 8$

$x = 269$

(6)

$10^2 \equiv 15 \pmod{17}$ square and reduce mod 17

$10^4 \equiv 4 \pmod{17}$

$10^8 \equiv 16 \pmod{17}$

$10^{16} \equiv 1 \pmod{17}$

$10^{32} \equiv 1 \pmod{17}$

$$10^{46} \equiv 10^{32} \cdot 10^8 \cdot 10^4 \cdot 10^2 \pmod{17}$$

$$\equiv 1 \cdot 16 \cdot 4 \cdot 15 \pmod{17}$$

$$\equiv 960 \pmod{17}$$

$$\equiv 8 \pmod{17}$$

7 a) $24310 = 2(10608) + 3094$
$10608 = 3(3094) + 1326$
$3094 = 2(1326) + 442$ ⇐ last nonzero remainder is the gcd
$1326 = 3(442) + 0$

$\gcd(24310, 10608) = 442$

b) First solve $442 = 24310a + 10608b$
then (multiplying by 3) $1326 = 24310(3a) + 10608(3b)$

$442 = 3094 - 2(1326)$ above
$442 = 3094 - 2[10608 - 3(3094)]$ above
$442 = 7(3094) - 2(10608)$ rewrite
$442 = 7[24310 - 2(10608)] - 2(10608)$ above
$442 = 7(24310) - 16(10608)$
$a = 7 \quad b = -16$

$1326 = 24310(21) + 10608(-48)$
$x = 21 \quad y = -48$

Even faster: Just use the first two lines of part a).
This gives another solution $x = -3 \quad y = 7$

(8) Basis Step: $P(1)$ $\quad 1^2 = \frac{1(2)(3)}{6}$ ✓

Induction Step $P(k)$: $\quad 1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$

$$\Rightarrow \quad 1^2 + 2^2 + \dots + (k+1)^2 = \left[1^2 + \dots + k^2\right] + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad (*)$$

$$= (k+1)\left[\frac{k(2k+1)}{6} + k+1\right]$$

$$= (k+1)\left[\frac{k(2k+1) + 6k + 6}{6}\right]$$

$$= (k+1)\frac{(2k^2 + 7k + 6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

This is $P(k+1)$ ✓

(9) Show that $3n^2 + 9n = 6m$ for some integer m $(n \geq 1)$

Basis Step: $P(1)$: $12 = 6(2)$ ✓

Induction Step $P(k)$: $3k^2 + 9k = 6m$ for some integer m.

$$
\begin{aligned}
\Rightarrow 3(k+1)^2 + 9(k+1) &= 3(k^2 + 2k + 1) + 9k + 9 \\
&= 3k^2 + 6k + 3 + 9k + 9 \\
&= \boxed{3k^2 + 9k} + 6k + 12 \\
&= 6m + 6(k+2) \quad (*) \\
&= 6(m + k + 2)
\end{aligned}
$$

where $m+k+2$ is an integer.

This is $P(k+1)$ ✓

(10) a) $a_0 = 3 \quad a_n = a_{n-1} + 2$ for $n \geq 1$

b) $a_1 = 7 \quad a_n = 7a_{n-1}$ for $n \geq 2$

c) $\boxed{a_1 = 2 \quad a_n = 7 - a_{n-1} \text{ for } n \geq 2}$

OR $\boxed{a_1 = 2 \quad a_2 = 5 \quad a_n = a_{n-2} \text{ for } n \geq 3}$