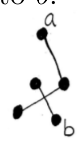
MORE ON GRAPH THEORY

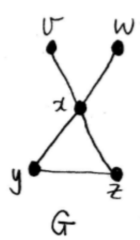
Section 1. Minimum Weight Spanning Trees

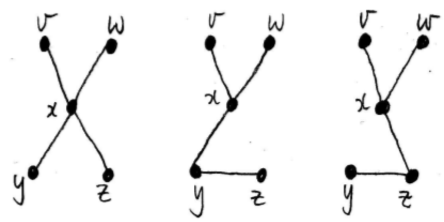
Recall that a graph is **connected** if there is at least one path between every pair of distinct vertices. Also recall that a **tree** is a connected graph with no cycles. A **spanning tree** of a connected graph G is a tree containing all the vertices of G whose edges are chosen from the edges of G.

Example: The graph below is not connected because there is no path from a to b.



Example: A graph G and the three possible spanning trees of G.



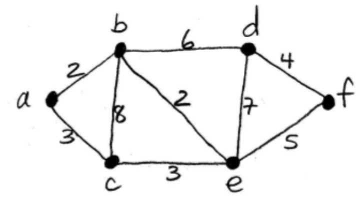


Given a connected graph G whose edges are weighted with a distance (or cost), we can find a minimum weight spanning tree of G using **Prim's Algorithm**. Below is an example of pseudocode for Prim's algorithm.

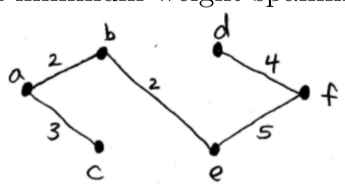
Find a minimum weight spanning tree in a graph G with n vertices

$T =$ a minimum-weight edge
for $i = 1$ to $i = n - 2$
begin
    $e =$ an edge of minimum weight incident to a vertex in T and not forming a cycle in T
    $T = T$ with e added
end
return(T)

Example: Let's perform Prim's algorithm on the graph below. We use alphabetical order to break ties between edges.



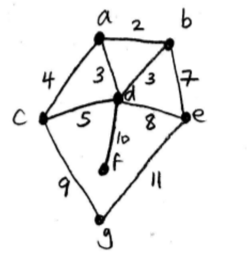
We will list the tree T as edges are added.
$T = \{ab\}$ (Choose a minimum-weight edge).
$T = \{ab, be\}$ (Choose a minimum-weight edge incident to a vertex in T).
$T = \{ab, be, ac\}$
$T = \{ab, be, ac, ef\}$ (The edge ce would create a cycle $abeca$.)
$T = \{ab, be, ac, ef, df\}$
A minimum weight spanning tree has weight 16. One is pictured below.



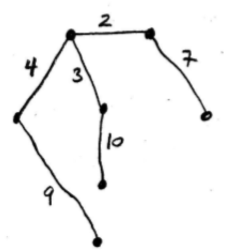
Other minimum weight spanning trees are possible in the above example if we don't break ties alphabetically. Using alphabetical order to break ties tends to be the easiest way to ensure that there is a single correct choice at each stage of the algorithm.

One application of Prim's Algorithm lies in computer and telecommunications networks. A minimum weight spanning tree minimizes the length of cable required to connect various points in the network.

Example: Let's perform Prim's algorithm on the graph below. We use alphabetical order to break ties between edges.



We will list the tree T as edges are added.
$T = \{ab\}$
$T = \{ab, ad\}$
$T = \{ab, ad, ac\}$
$T = \{ab, ad, ac, be\}$
$T = \{ab, ad, ac, be, cg\}$
$T = \{ab, ad, ac, be, cg, df\}$
A minimum weight spanning tree has weight 35. One is pictured below.

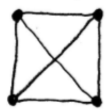


Recall that a **greedy algorithm** is an algorithm that makes the best choice at each stage of the problem. Prim's algorithm is greedy because when it examines the eligible edges it always chooses one of minimum weight. The minimum weight spanning tree problem is an example of a problem for which the greedy algorithm provides an optimal solution.

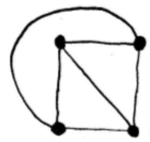
Section 2. Planar Graphs

A graph is **planar** if it can be embedded in the plane so that none of its edges cross.

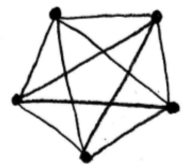
Example 1: The complete graph K_4 is drawn below. It is drawn first with crossing edges, and then it is drawn embedded in the plane. The complete graph K_5 is nonplanar; K_5 cannot be drawn without any of its edges crossing.



K_4



K_4



K_5

Given a planar embedding of a graph, let F represent the number of regions (faces) the plane is divided into. The number of faces includes the one infinite region on the exterior of the graph. Let V and E represent the number of vertices and edges in the graph.

Example 2: A planar embedding of the complete bipartite graph $K_{2,3}$ is drawn below. For the graph below, $F = 3, V = 5$ and $E = 6$.

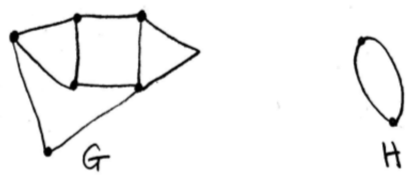
$K_{2,3}$

Euler's Formula A connected planar graph satisfies $V - E + F = 2$.

Example 3: Euler's Formula holds for the graph $K_{2,3}$ since $5 - 6 + 3 = 2$. Euler's Formula also holds for the graph K_4 in Example 1 since $4 - 6 + 4 = 2$.

Given a planar embedding of a graph, the **degree** of a face is the number of edges that bound it. For instance, a triangular face has degree 3. Let F_i represent the number of faces of degree i, for $i \geq 2$.

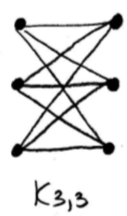
Example 4: The planar embedding of K_4 in Example 1 has four faces of degree 3. So $F_3 = 4$ and $F_i = 0$ for $i \neq 3$. The planar embedding of $K_{2,3}$ in Example 2 has three faces of degree 4. So $F_4 = 3$ and $F_i = 0$ for $i \neq 4$. The planar embedding of the graph G below has $F_2 = 0, F_3 = 2, F_4 = 2, F_5 = 0, F_6 = 1$ and $F_i = 0$ for $i > 6$. The planar embedding of the graph H below has $F_2 = 2$ and $F_i = 0$ for $i \neq 2$. A face of degree 2 is only possible if multiple edges are permitted in the graph.



The Edge Formula For any planar embedding $2F_2 + 3F_3 + 4F_4 + \ldots = 2E$.
Proof: For each face, count the number of edges that bound it. There are F_2 regions of degree 2, each bounded by two edges. Similarly, there are F_i regions of degree i, each bounded by i edges for $i \geq 3$. The sum counts each edge in the graph twice since each edge lies on the boundary of exactly two faces.

Example 5: Consider the graph G in Example 4. The graph G has $E = 10$. The Edge Formula holds because $2(0) + 3(2) + 4(2) + 5(0) + 6(1) = 2(10)$.

Consider the problem of joining three houses to three utilities (say gas, water and sewage). Is it possible to join the houses and utilities so that none of the connections cross? We are asking whether the complete bipartite graph $K_{3,3}$ below is planar.



Theorem: The graph $K_{3,3}$ is nonplanar.

Feel free to just skim the proof below. It gives a good overview of some of the techniques in planar graph theory.

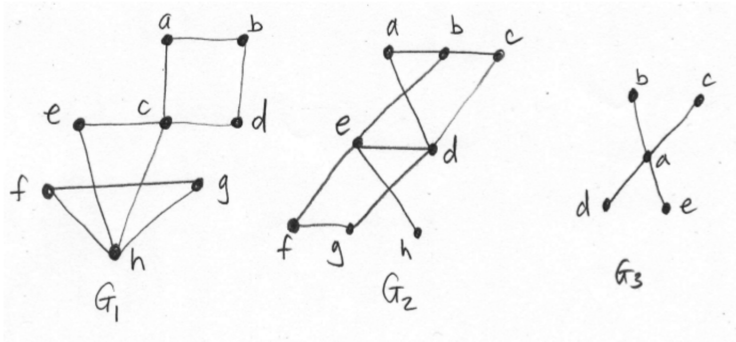
Proof: Suppose for contradiction that $K_{3,3}$ has a planar embedding. There are no cycles of length three in $K_{3,3}$, nor are there multiple edges in the graph. This means there are no faces of degree two or three. The Edge Formula for $K_{3,3}$ becomes $4F_4 + 5F_5 + \ldots = 2E$. This implies $4(F_4 + F_5 + \ldots) \leq 2E$. Recalling that F represents the total number of faces in the embedding, we have $4F \leq 2E$. This implies $F \leq \frac{1}{2}E$. Substituting $F \leq \frac{1}{2}E$ into Euler's Formula gives $V - E + \frac{1}{2}E \geq 2$, or $V - \frac{1}{2}E \geq 2$. But the graph $K_{3,3}$ has $V = 6$ and $E = 9$, contradicting this inequality. Therefore $K_{3,3}$ does not have a planar embedding.

Planar graphs are important in the design of electronic circuits. A circuit can be printed on a single board with no connections crossing exactly when the graph describing the circuit is planar. When the graph is not planar, the minimum number of layers needed to construct the circuit is calculated. This minimum number is called the **thickness** of the graph.

Section 3. Euler Trails and Circuits

An **Euler circuit** traverses each edge of a graph exactly once and returns to the starting vertex. An **Euler trail** traverses each edge of a graph exactly once, but begins and ends at different vertices.

Example 1: The sequence of edges $cabdcehfghc$ is an Euler circuit in G_1 below. The sequence of edges $hedgfebadcb$ is an Euler trail in G_2 below. The graph G_3 has neither an Euler circuit nor an Euler trail.

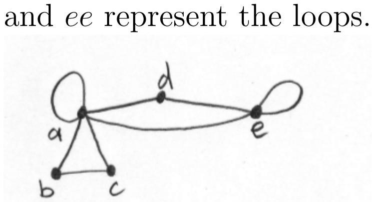


Recall that the **degree** of a vertex is the number of edges incident with it. For example, the list of degrees in G_1 is $2, 2, 4, 2, 2, 2, 2, 4$.

Theorem A graph has an Euler circuit if and only if each vertex has even degree. A graph has an Euler trail if and only if there are exactly two vertices of odd degree in the graph.

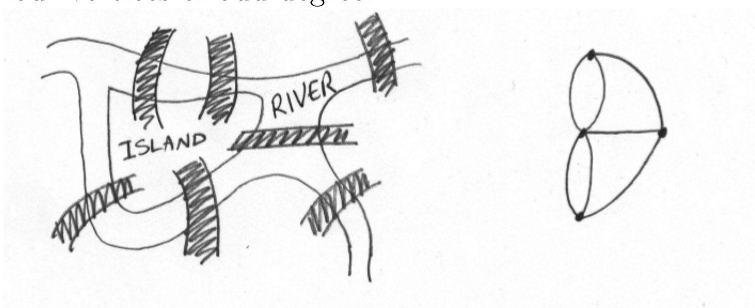
Example 2: Looking back at Example 1, G_1 has an Euler circuit because all its vertices have even degree. Because the graph G_2 has exactly two vertices of odd degree (b and h), G_2 has an Euler trail beginning and ending at these two vertices. The graph G_3 has neither an Euler circuit nor an Euler trail because it has four vertices of odd degree.

Example 3: If a graph contains loops, each loop contributes 2 to the degree of the vertex. The list of degrees in the graph below is $6, 2, 2, 2, 4$. Let's find an Euler circuit in the graph. One possible circuit is $aabcaeeda$, where aa and ee represent the loops.

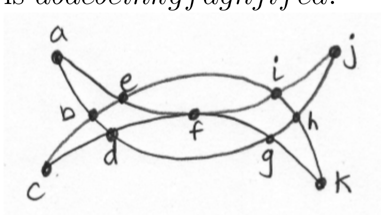


Example 4: The Seven Bridges of Königsberg.

In the 1700's the town of Königsberg, Prussia had seven bridges as depicted below. The townspeople wondered whether it was possible to cross each bridge exactly once and return to the starting point. The famous mathematician Leonhard Euler answered this question in 1736–his solution was possibly the invention of graph theory. Euler modelled the town as a graph, with each bridge represented by an edge. The question became: does the graph contain an Euler circuit? The answer is no, since the graph has four vertices of odd degree.



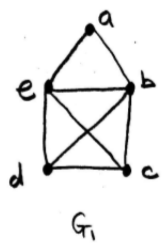
Example 5: Find an Euler circuit in the graph below. Intuitively, we could start tracing out one crescent, jump over to the second crescent and trace it out completely, and then finish tracing out the first crescent. One example is $abdcbeihkgfdghjifea$.

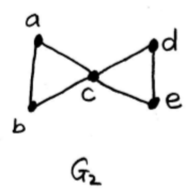


Section 4. Hamilton Paths and Cycles

A **Hamilton cycle** is a cycle that passes through each vertex of a graph exactly once. A **Hamilton path** passes through each vertex of a graph exactly once, but begins and ends at different vertices.

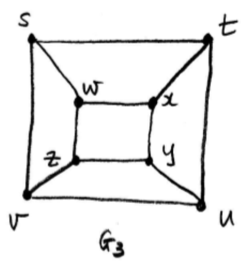
Example: The graph G_1 below has a Hamilton cycle *abcdea*. The graph G_2 has a Hamilton path *abcde*, but it has no Hamilton cycle.



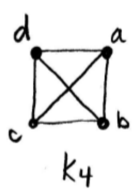


There is no known condition that determines exactly which graphs have Hamilton cycles. We will focus on a few typical examples.

Example: Find a Hamilton cycle in the graph G_3 below. Intuitively, we could travel around most of the inside cycle, then jump to the outside cycle and travel around it before finishing back on the inside cycle. For example *wxyzvutsw* is a Hamilton cycle in G_3.



Example: How many different Hamilton cycles does the complete graph K_4 have? Recall that every pair of distinct vertices in K_4 are connected by an edge. One cycle is considered different from another if it visits the vertices in a different order. For example, the cycle *abcda* is different than *adcba*, but *abcda* and *bcdab* are the same cycle written in different ways.

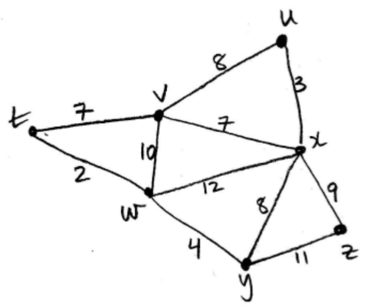


Solution: We must visit the vertex a at some point on the cycle. There are 3 choices for the next vertex, then 2 choices for the next one, then 1 choice for the last vertex before the cycle returns to vertex a. All together, there are $3 \times 2 \times 1 = 6$ possible Hamilton cycles in K_4.

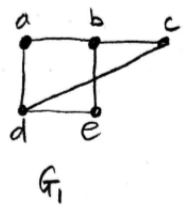
The complete graph K_n has $(n-1)!$ different Hamilton cycles, for $n \geq 3$.

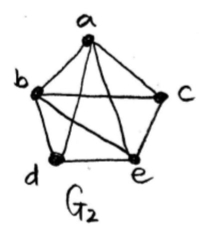
Section 5. Exercises

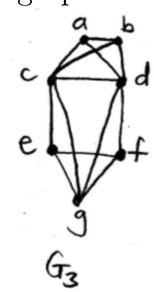
1. Use Prim's algorithm to find a minimum weight spanning tree for the graph below. Write out the edges of your tree after each step of the algorithm.



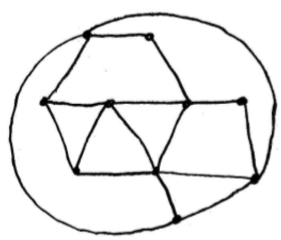
2. Find a planar embedding for each planar graph below.



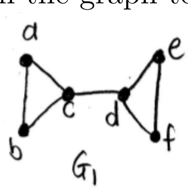


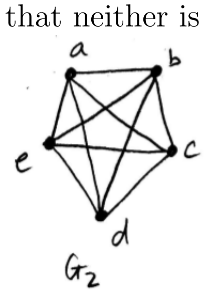


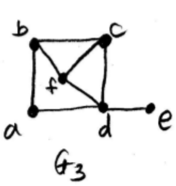
3. Verify Euler's Formula and the Edge Formula for the planar embedding below.

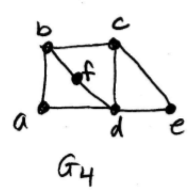


4. Find an Euler trail or Euler circuit in each graph below, or list the degrees in the graph to show that neither is possible.









5. For each graph below, find a Hamilton cycle if one exists. If there is no Hamilton cycle, find a Hamilton path or state that one does not exist.