Final Exam Review

The first two problems will be especially good review of Assignment 3.

① Set $A = \{a, b, c, d, e, f, g\}$
A program prints 290 subsets of A.
Show that some subset is printed ≥ 3 times.

Solution: $n = 290$ printed subsets
$m = 2^7 = 128$ different subsets of A.

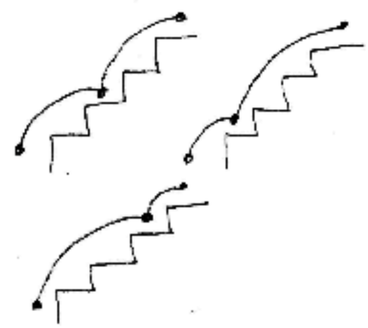
By the Pigeonhole Principle, at least one subset is printed $\geq \left\lceil \frac{290}{128} \right\rceil$
≥ 3 times.

② For $n \geq 2$, let a_n = # ways to climb n steps if you can only climb 2 or 3 steps at a time.

Solution:

$a_2 = 1$

$a_3 = 1$



$a_4 = 3$
3 different ways to climb 4 steps

To climb n steps:

Climb $n-2$

or

Climb $n-3$

ways is $a_n = a_{n-2} + a_{n-3}$

$$\boxed{\begin{array}{l} a_n = a_{n-2} + a_{n-3}\ , \quad n \geq 5 \\ a_2 = 1 \quad a_3 = 2 \quad a_4 = 3 \end{array}}$$

③ Domain of x, y: integers

$P(x,y)$: $x = y^2$

$Q(x,y)$: $x + 5 \geq 2 - y$

State truth values of:

a) $P(-1,1)$ FALSE

b) $P(2,-3) \vee Q(2,-3)$ $F \vee T =$ TRUE

c) $\exists y\ P(3,y)$ FALSE

d) $\forall x\ \exists y\ Q(x,y)$ TRUE

For each x there is a y that works.

e) $\exists x\ \forall y\ Q(x,y)$ FALSE

There is an x so that for all y $\Leftarrow$ false

(4) Let n be an integer.
Prove: $23n^3+8$ odd $\Rightarrow$ n odd.

Proof: By contrapositive [n even $\Rightarrow 23n^3+8$ even.]
Let $n=2k$, k an integer.

$$\Rightarrow 23n^3+8 = 23(2k)^3+8$$
$$= 23(8k^3)+8$$
$$= 2[92k^3+4], \quad 92k^3+4 \text{ an integer}$$
$$\Rightarrow 23n^3+8 \text{ even.}$$

(5) How many subsets of $A=\{2,3,4,\dots,9\}$ contain 4 but not 8?

Solution: Freedom to include/not include 2, 3, 5, 6, 7, 9
subsets $= 2^6$

(6) Let $f(x) = \lfloor x \rfloor + \lceil x \rceil$
be a function from the real numbers to the integers

a) Is $f(x)$ one-to-one?
No. $f(0.1) = 1 = f(0.2)$

b) Is $f(x)$ onto?
Yes. Every integer "gets hit"!

$$\uparrow$$
$$f(-0.1) = -1$$
$$f(0) = 0$$
$$f(0.1) = 1$$
$$f(1) = 2$$
$$f(1.1) = 3$$
$$\downarrow$$

(7) a) Show that $f(n) = \frac{n^6 + 5n^3}{n^2+1}$ is $O(n^4)$.

Solution: $|f(n)| \le \frac{n^6 + 5n^3}{n^2+1}$ for $n > 0$

$\le \frac{n^6 + 5n^6}{n^2+1}$ for $n > 1$

$\le \frac{6n^6}{n^2}$ for $n > 1$

$\le 6n^4$ for $n > 1$

$|f(n)| \le 6|n^4|$ for $n > 1$

$\Rightarrow f(n)$ is $O(n^4)$.

b) Is $g(n) = 1+2+3+ \ldots + n$ $O(n)$?

No. From class, $g(n) = 1+2+\ldots+n$

$= \frac{n(n+1)}{2}$

$g(n)$ is $O(n^2)$, but not $O(n)$.

(8) Compute $500^{11} \pmod{601}$

Solution:

$500^2 \equiv 585 \pmod{601}$

$500^4 \equiv 256 \pmod{601}$

$500^8 \equiv 27 \pmod{601}$

square and reduce mod 601

$500^{11} = 500^8 \cdot 500^2 \cdot 500^1$

$\equiv 27 \cdot 585 \cdot 500 \pmod{601}$

$\equiv 360 \pmod{601}$

(9) a) Find gcd (111, 297)

$$297 = 2(111) + 75$$
$$111 = 1(75) + 36$$
$$75 = 2(36) + 3 \Leftarrow \text{last nonzero remainder}$$
$$36 = 12(3) + 0$$

$$\gcd(111, 297) = 3$$

b) Write the gcd as $111a + 297b$

$$3 = 75 - 2(36)$$
$$3 = 75 - 2[111 - 1(75)]$$
$$3 = 3(75) - 2(111)$$
$$3 = 3[297 - 2(111)] - 2(111)$$
$$3 = 3(297) - 8(111)$$

(10) Use induction to show that for $n \geq 1$:
$$1 + 3 + 5 + \dots + (2n-1) = n^2$$

Basis Step: $(n=1)$ $\quad 1 = 1^2$ ✓

Induction Step: Let $k \geq 1$
$$1 + 3 + \dots + (2k-1) = k^2$$
$$\Rightarrow 1 + 3 + \dots + (2(k+1)-1) = [1 + \dots + (2k-1)] + (2k+1)$$
$$= k^2 + 2k + 1$$
$$= (k+1)^2$$ ✓

(11) Give a recursive definition for the sequence $a_0, a_1, a_2, \dots$ given by $4, 7, 13, 22, 34, 49, \dots$

$$\boxed{\begin{array}{c} a_n = a_{n-1} + 3n \ , \quad n \geq 1 \\ a_0 = 4 \end{array}}$$

(12) Find the coefficient of $x^{10}y^4$ in the expansion of $(2x-5y)^{14}$

Solution: By the binomial theorem:

$$(a+b)^{14} = \sum_{n=0}^{14} \binom{14}{n} a^{14-n} b^n$$

$$\Rightarrow (2x-5y)^{14} = \sum_{n=0}^{14} \binom{14}{n} (2x)^{14-n} (-5y)^n$$

when $n=4$ we get $\binom{14}{4}(2x)^{10}(-5y)^4$

correct powers ✓

The coefficient is $\binom{14}{4}(2)^{10}(-5)^4$

$$= \binom{14}{4} 2^{10} \cdot 5^4$$

or $14C4 \cdot 2^{10} \cdot 5^4$

(13) How many ways are there to choose 18 doughnuts from 5 different types?

18 $*$'s
4 $|$'s
22 objects

22C4 ways.

(14) Solve $a_n = 6a_{n-1} - 8a_{n-2} + 2\cdot 3^n$, $n \geq 2$
$a_0 = -15 \quad a_1 = -44$

$r^2 = 6r - 8$
$r^2 - 6r + 8 = 0$
$(r-2)(r-4) = 0$
$r = 2, 4$
$a_n^{(h)} = A\cdot 2^n + B\cdot 4^n$

$a_n^{(p)} = C\cdot 3^n$

$a_n = 6a_{n-1} - 8a_{n-2} + 2\cdot 3^n$
$C\cdot 3^n = 6(C\cdot 3^{n-1}) - 8(C\cdot 3^{n-2}) + 2\cdot 3^n$
$\div 3^{n-2} \quad 9C = 18C - 8C + 2\cdot 9$
$-C = 18$
$C = -18$

Now $a_n = a_n^{(h)} + a_n^{(p)}$
$a_n = A\cdot 2^n + B\cdot 4^n - 18\cdot 3^n$

$\rightarrow$

$a_0 = -15$:

$-15 = A + B - 18 \qquad A + B = 3$

$a_1 = -44$

$-44 = 2A + 4B - 54 \qquad 2A + 4B = 10$

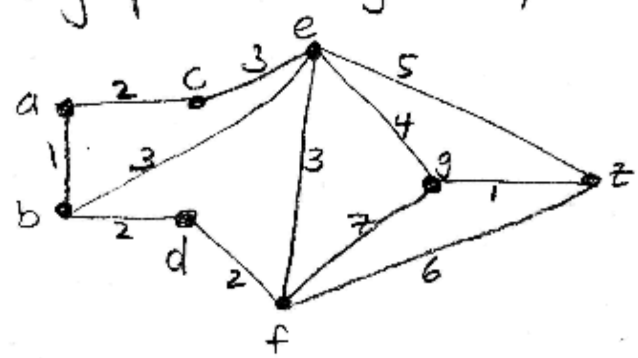
$\left.\begin{matrix} A+B=3 \\ 2A+4B=10 \end{matrix}\right\} \begin{matrix} A=1 \\ B=2 \end{matrix}$

$$\boxed{a_n = 2^n + 2 \cdot 4^n - 18 \cdot 3^n, \quad n \geq 0}$$

(15) State the inclusion/exclusion formula for 3 sets.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

(16) The graph G is given by:



a) List the degrees of all vertices:

2, 3, 2, 2, 5, 4, 3, 3 (in alphabetical order)

b) Use Dijkstra's Algorithm to find the length of the shortest path from a to z.

	$\mathcal{L}(a)$	$\mathcal{L}(b)$	$\mathcal{L}(c)$	$\mathcal{L}(d)$	$\mathcal{L}(e)$	$\mathcal{L}(f)$	$\mathcal{L}(g)$	$\mathcal{L}(z)$
	0	∞	∞	∞	∞	∞	∞	∞
$S=\{a\}$		1	2	∞	∞	∞	∞	∞
$S=\{a,b\}$			2	3	4	∞	∞	∞
$S=\{a,b,c\}$				3	4	∞	∞	∞
$S=\{a,b,c,d\}$					4	5	∞	∞
$S=\{a,b,c,d,e\}$						5	8	9
$S=\{a,b,c,d,e,f\}$							8	9.
$S=\{a,b,c,d,e,f,g\}$								9

$S=\{a,b,c,d,e,f,g,z\}$ $\qquad \mathcal{L}(z)=9$

Length of shortest path $a \to z$ is 9.