Math 222 Assignment Three

Name:____________________________

Assignments must be completed on this paper. Marks may be deducted for not showing all your work.

1. [3 marks] Sixteen students take a quiz. The possible marks are the integers $0 - 15$. If the sixteen marks sum to 100, prove that at least two students got the same mark.

The 16 marks cannot all be different.
If they were, the marks would sum to
$0+1+2+\dots+15 = \frac{15(16)}{2}$, which is 120.

So Boxes: the different marks $m \le 15$
Objects: the 16 students $n=16$

By the Pigeonhole Principle, at least one box contains $\geq \lceil \frac{16}{15} \rceil = 2$ objects

$\Rightarrow$ At least two students got the same mark.

Alternatively: Proof by contradiction using first three lines.

2. [4 marks] Let m, r and j be nonnegative integers. Use the formula $nCk = \frac{n!}{k!(n-k)!}$ to prove that $mCr \cdot rCj = mCj \cdot (m-j)C(r-j)$.

$$LHS = mCr \cdot rCj$$

$$= \frac{m!}{\cancel{r!}(m-r)!} \cdot \frac{\cancel{r!}}{j!(r-j)!}$$

$$= \frac{m!}{j!} \cdot \frac{1}{(r-j)!(m-r)!}$$

$$= \frac{m!}{j!(m-j)!} \cdot \frac{(m-j)!}{(r-j)!(m-r)!}$$

$$= mCj \cdot (m-j)C(r-j)$$

$$= RHS$$

3. [5 marks] a) State the Binomial Theorem.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

b) Use part a) to find a simpler expression for $\sum_{k=0}^{n} \binom{n}{k} 2^k$.

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 2^k$$

$$= (1+2)^n$$

$$= 3^n$$

c) Confirm your answer to part b) using the value $n = 6$.

Confirm $\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$ with $n=6$

$$LHS = \binom{6}{0}2^0 + \binom{6}{1}2^1 + \binom{6}{2}2^2 + \binom{6}{3}2^3 + \binom{6}{4}2^4 + \binom{6}{5}2^5 + \binom{6}{6}2^6$$

$$= 1 + 6 \cdot 2 + 15 \cdot 4 + 20 \cdot 8 + 15 \cdot 16 + 6 \cdot 32 + 1 \cdot 64$$

$$= 729$$

$$= 3^6$$

$$= RHS \checkmark$$

4. [4 marks] Find the number of integer solutions to the following equation if $x_1, x_3, x_5 \geq 0, x_2 \geq -5$ and $x_4 \geq 2$: $x_1 + x_2 + x_3 + x_4 + x_5 = 21$.

Let $z_2 = x_2 + 5 \geq 0$ $z_4 = x_4 - 2 \geq 0$

Then $x_1 + z_2 + x_3 + z_4 + x_5 = x_1 + x_2 + x_3 + x_4 + x_5 + 3$
$= 24$

with x_1, z_2, x_3, z_4, x_5 non-negative.

\# solutions = # ways to arrange 24 *'s
and 4 |'s
$\underline{}$
28 objects

$= 28C4$
$= 20,475$

5. [5 marks] We are rolling a four-sided die (with sides labelled 1, 2, 3, 4). Find a recurrence relation for the number of ways the rolls can sum to n for $n \geq 1$. (For example, here are the ways in which the rolls can sum to 3: 1-1-1, 1-2, 2-1, 3).

Let a_n = # ways the rolls can sum to n

a_n : $\boxed{1}$ $\boxed{\quad\uparrow\quad}$
$1 \times a_{n-1}$ ways to sum to $n-1$

$\boxed{2}$ $\boxed{\quad\uparrow\quad}$
$1 \times a_{n-2}$ ways

$\boxed{3}$ $\boxed{\quad\uparrow\quad}$
$1 \times a_{n-3}$ ways

$\boxed{4}$ $\boxed{\quad\uparrow\quad}$
$1 \times a_{n-4}$ ways

$$\boxed{a_n = a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}, \quad n \geq 5}$$

$a_1 = 1$: $\boxed{1}$

$a_2 = 2$: $\boxed{1}\boxed{1}$, $\boxed{2}$

$a_3 = 4$: $\boxed{1}\boxed{1}\boxed{1}$, $\boxed{2}\boxed{1}$, $\boxed{1}\boxed{2}$, $\boxed{3}$

$a_4 = 8$: $\boxed{1}\boxed{1}\boxed{1}\boxed{1}$, $\boxed{1}\boxed{1}\boxed{2}$, $\boxed{1}\boxed{2}\boxed{1}$, $\boxed{2}\boxed{1}\boxed{1}$, $\boxed{2}\boxed{2}$, $\boxed{1}\boxed{3}$, $\boxed{3}\boxed{1}$, $\boxed{4}$

6. [6 marks] Solve the following recurrence relation:
$a_n = 10a_{n-1} - 25a_{n-2} + 4 \cdot 3^n$ for $n \geq 2$, $a_0 = 11$, $a_1 = 17$.

$r^2 = 10r - 25$
$r^2 - 10r + 25 = 0$
$(r-5)^2 = 0$
$r = 5$

$a_n^{(h)} = A5^n + Bn5^n$

nonhomogeneous part is $4 \cdot 3^n$

$a_n^{(p)} = C \cdot 3^n$

Use recurrence to find C:

$C \cdot 3^n = 10(C \cdot 3^{n-1}) - 25(C \cdot 3^{n-2}) + 4 \cdot 3^n$

$\div 3^{n-2}$ $\quad 9C = 30C - 25C + 4 \cdot 9$

$4C = 36$

$C = 9$

$a_n^{(p)} = 9 \cdot 3^n$ or 3^{n+2}

Now $a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = (A + Bn)5^n + 3^{n+2}$

Use initial conditions

$a_0 = 11$: $\quad 11 = A + 9 \quad \Rightarrow \quad A = 2$

$a_1 = 17$: $\quad 17 = (2+B)5 + 27 \quad \Rightarrow \quad 17 = 10 + 5B + 27$

$-20 = 5B$

$B = -4$

$$\boxed{a_n = (2 - 4n)5^n + 3^{n+2}, \quad n \geq 0}$$

7. [3 marks] a) State the inclusion/exclusion formula for three sets A, B, C.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

b) Use the formula to find $|A \cup B|$ if $|A \cup B \cup C| = 80$, $|A| = 56$, $|B| = 57$, $|C| = 59$, $|A \cap C| = 46$, $|B \cap C| = 44$ and $|A \cap B \cap C| = 41$.

First, find $|A \cap B|$ Then $|A \cup B| = |A| + |B| - |A \cap B|$

Plugging into the formula in a),

$$80 = 56 + 57 + 59 - |A \cap B| - 46 - 44 + 41$$

$$-43 = -|A \cap B|$$

$$|A \cap B| = 43$$

Now $|A \cup B| = |A| + |B| - |A \cap B|$

$$= 56 + 57 - 43$$

$$= 70$$