Math 222 Assignment Two

Name:________________________________

Assignments must be completed on this paper. Marks may be deducted for not showing all your work.

1. [3 marks] Show that $(3n+7)\log_2(n^2+1)$ is $O(n\log_2 n)$.

$$
\begin{aligned}
|(3n+7)\log_2(n^2+1)| &\le (3n+7)\log_2(n^2+1) && \text{for } n>0 \\
&\le (3n+7n)\log_2(n^2+1) && \text{for } n>1 \\
&\le 10n\log_2(n^2+n^2) && \text{for } n>1 \\
&\le 10n\log_2(2n^2) && \text{for } n>1 \\
&\le 10n\log_2(n^3) && \text{for } n>2 \\
&\le 10n\cdot 3\cdot\log_2 n && \text{for } n>2 \\
&\le 30n\log_2 n && \text{for } n>2
\end{aligned}
$$

$\Rightarrow \quad |(3n+7)\log_2(n^2+1)| \le 30|n\log_2 n| \quad$ for $n>2$

$\Rightarrow \quad (3n+7)\log_2(n^2+1)$ is $O(n\log_2 n)$

2. [4 marks] a) Write 4599504 as a product of prime powers.
(For example $2646 = 2 \times 3^3 \times 7^2$).

$2^4 \mid 4599504$ $\qquad$ $4599504 = 2^4 \cdot 3^5 \cdot 7 \cdot 13^2$

$3^5 \mid 4599504$

$7 \mid 4599504$ $\qquad$ $13^2 \mid 4599504$

b) Write 907500 as a product of prime powers.

$2^2 \mid 907500$ $\qquad$ $11^2 \mid 907500$ $\qquad$ $907500 = 2^2 \cdot 3 \cdot 5^4 \cdot 11^2$

$3 \mid 907500$

$5^4 \mid 907500$

c) Write gcd(4599504, 907500) as a product of prime powers.

$\gcd(4599504, 907500) = 2^2 \cdot 3 \leftarrow$ use the minimum of the powers for each prime

3. [3 marks] Compute 9^{39} (mod 11) using successive squaring. Show all your work.

$9^2 \equiv 4 \pmod{11}$

$9^4 \equiv 5 \pmod{11}$ $\qquad$ square and reduce mod 11

$9^8 \equiv 3 \pmod{11}$

$9^{16} \equiv 9 \pmod{11}$

$9^{32} \equiv 4 \pmod{11}$

$$9^{39} \equiv 9^{32} \cdot 9^4 \cdot 9^2 \cdot 9^1 \pmod{11}$$
$$\equiv 4 \cdot 5 \cdot 4 \cdot 9 \pmod{11}$$
$$\equiv 720 \pmod{11}$$
$$\equiv 5 \pmod{11}$$

4. [6 marks] a) Use the Euclidean Algorithm to find gcd(1545, 240).

$$1545 = 6(240) + 105$$
$$240 = 2(105) + 30$$
$$105 = 3(30) + 15 \leftarrow$$ last nonzero remainder is the gcd
$$30 = 2(15) + 0$$

$$\gcd(1545, 240) = 15$$

b) Use part a) to find integers x and y so that $165 = 1545x + 240y$.

First, solve $15 = 1545a + 240b$
then $165 = 1545(11a) + 240(11b)$
(multiply by 11)

$$15 = 105 - 3(30)$$ above
$$15 = 105 - 3[240 - 2(105)]$$ above
$$15 = 7(105) - 3(240)$$ rewrite
$$15 = 7[1545 - 6(240)] - 3(240)$$ above
$$15 = 7(1545) - 45(240)$$ rewrite

$a = 7 \quad b = -45$

$$165 = 1545(77) + 240(-495)$$

$x = 77 \quad y = -495$

5. [4 marks] Use mathematical induction to show that $1^3+2^3+\ldots+n^3=[\frac{n(n+1)}{2}]^2$ for $n \geq 1$.

Basis Step $P(1)$: $1^3 = \left[\frac{1(2)}{2}\right]^2$

Induction step: Let $k \geq 1$ $P(k)$: $1^3+2^3+\ldots+k^3 = \left[\frac{k(k+1)}{2}\right]^2$

$$\Rightarrow 1^3+2^3+\ldots+(k+1)^3 = 1^3+\ldots+k^3+(k+1)^3$$

$$= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 \quad (*)$$

$$= (k+1)^2\left[\frac{k^2}{4}+k+1\right]$$

$$= (k+1)^2\left[\frac{k^2+4k+4}{4}\right]$$

$$= (k+1)^2\,\frac{(k+2)^2}{4}$$

$$= \left[\frac{(k+1)(k+2)}{2}\right]^2$$

This is $P(k+1)$

6. [4 marks] Use mathematical induction to show that
$(1-\frac{1}{2})(1-\frac{1}{3})\cdot\ldots\cdot(1-\frac{1}{n+1}) = \frac{1}{n+1}$ for $n \geq 1$.

Basis Step $P(1)$: $\left(1-\frac{1}{2}\right) = \frac{1}{2}$ ✓

Induction Step: $P(k)$: Let $k \geq 1$ $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k+1}\right) = \frac{1}{k+1}$

$$\Rightarrow \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdot\ldots\cdot\left(1-\frac{1}{k+2}\right) = \left[\left(1-\frac{1}{2}\right)\cdot\ldots\cdot\left(1-\frac{1}{k+1}\right)\right]\left(1-\frac{1}{k+2}\right)$$

$$= \frac{1}{k+1}\cdot\left(1-\frac{1}{k+2}\right) \quad (*)$$

$$= \frac{1}{k+1}\left(\frac{k+2-1}{k+2}\right)$$

$$= \frac{1}{k+1}\cdot\frac{k+1}{k+2}$$

$$= \frac{1}{k+2}$$

This is $P(k+1)$. ✓

7. [6 marks] Give a recursive definition for each sequence below.

a) $a_n = 3(5n+1) - 8$ for $n \geq 0$

Rewrite: $a_n = 15n - 5$

$\boxed{a_0 = -5 \quad a_n = a_{n-1} + 15 \text{ for } n \geq 1}$

b) the sequence $a_1, a_2, a_3, a_4, \ldots$ given by $-3, 4, -3, 4, \ldots$

$\boxed{a_1 = -3 \quad a_n = 1 - a_{n-1} \text{ for } n \geq 2}$

OR: $\boxed{a_1 = -3,\ a_2 = 4 \quad a_n = a_{n-2} \text{ for } n \geq 3}$

c) the sequence $a_1, a_2, a_3, a_4, a_5, a_6, \ldots$ given by $1, 3, 7, 15, 31, 63, \ldots$

double and add 1

$\boxed{a_1 = 1 \quad a_n = 2a_{n-1} + 1 \text{ for } n \geq 2}$

OR: $\boxed{a_1 = 1 \quad a_n = a_{n-1} + 2^{n-1} \text{ for } n \geq 2}$