6. Continuous Random Variables

<u>Discrete random variable</u>: has gaps between possible values

<u>Continuous random variable</u>: no gaps between possible values

Alternatively: A continuous random variable has ∞-many decimal places

Examples of D.R.V.:

1's that appear in 20 dice rolls

of visits to a website next week

(Possible values are 0, 1, 2, ...)

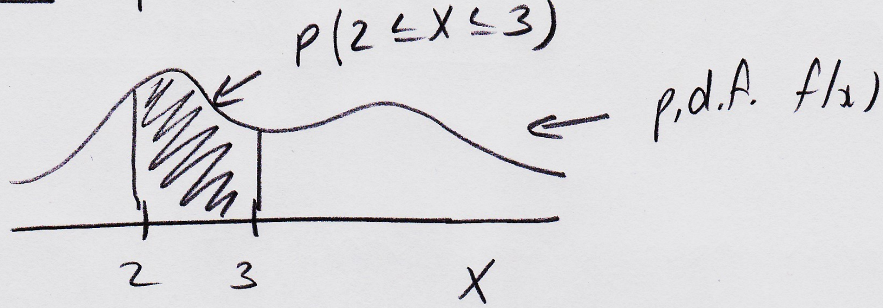
Examples of C.R.V.:

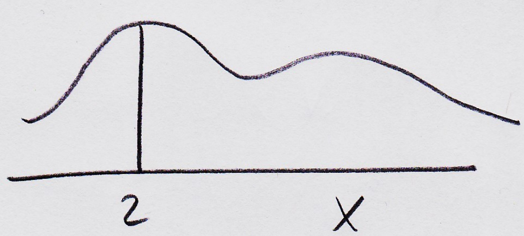
length (could be 1.26897... m)

mass

If X is a C.R.V. then probability is defined as area under a curve. The curve is called the <u>probability density function</u> (p.d.f.)

<u>Ex</u>:



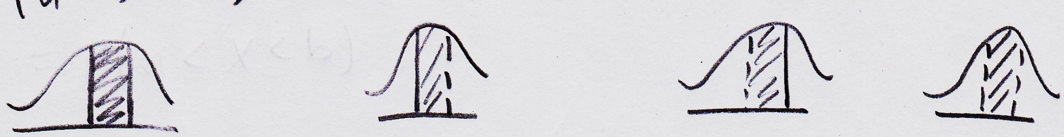


$P(X=2)=0$ because a line has zero area

If X is a C.R.V. :

1) $P(X=c)=0$ for any # c

2) $P(a \le X \le b) = P(a \le X < b) = P(a < X \le b) = P(a < X < b)$



Ex: The p.d.f. for X is

$$f(x) = \begin{cases} x/8 , & 0 < x \le 2 \\ 1/4 ; & 2 < x \le 5 \\ 0 ; & \text{otherwise} \end{cases}$$

Find :

a) $P(X = 2.2)$

$= 0$

b) $P(1 \le X \le 3)$

$= \int_1^3 f(x)\,dx$

$= \int_1^2 \frac{x}{8}\,dx + \int_2^3 \frac{1}{4}\,dx$

$= \left[\frac{x^2}{16}\right]_1^2 + \left[\frac{x}{4}\right]_2^3$

$= \left(\frac{4}{16} - \frac{1}{16}\right) + \left(\frac{3}{4} - \frac{2}{4}\right)$

$= 0.4375$

c) $P(1 < X < 3)$

$= P(1 \le X \le 3)$

$= 0.4375$

d) $P(X > 1.2)$

$= \int_{1.2}^{\infty} f(x)\,dx$

$= \int_{1.2}^{2} \frac{x}{8}\,dx + \int_{2}^{5} \frac{1}{4}\,dx + \int_{5}^{\infty} 0\,dx$

$= \left[\frac{x^2}{16}\right]_{1.2}^{2} + \left[\frac{x}{4}\right]_{2}^{5}$

$= \left(\frac{4}{16} - \frac{1.2^2}{16}\right) + \left(\frac{5}{4} - \frac{2}{4}\right)$

$= 0.91$

e) $P(X < 0.6)$

$= \int_{-\infty}^{0.6} f(x)\,dx$

$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{0.6} \frac{x}{8}\,dx$

$= 0 + \left[\frac{x^2}{16}\right]_{0}^{0.6}$

$= \frac{0.6^2}{16}$

$= 0.0225$

A p.d.f. must satisfy:

1) Total area under curve = 1

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

2) $f(x) \geq 0$ for all x

<u>Ex</u>: Find the value of k that makes $f(x)$ a valid p.d.f.

$$f(x) = \begin{cases} kx^7, & 1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{1}^{2} kx^7\,dx = 1$$

$$\left[\frac{kx^8}{8}\right]_1^2 = 1$$

$$\frac{256k}{8} - \frac{k}{8} = 1$$

$$\frac{255}{8}k = 1$$

$\times \frac{8}{255}$: $\quad k = \frac{8}{255}$

Check that $f(x) \geq 0$ for all x ✓

<u>Ex</u>: The daily consumption of electricity (in millions of kilowatt hours) for a certain city has p.d.f.

$$f(x)=\begin{cases} \frac{x}{9}e^{-x/3}, & x>0 \\ 0, & \text{otherwise} \end{cases}$$

If the city's supply is 12 million kWh, what is the probability that the supply is inadequate on any given day?

Want $P(X>12)$

$$= 1 - P(X \le 12)$$

$$= 1 - \int_{-\infty}^{12} f(x)\,dx$$

$$= 1 - \int_{-\infty}^{0} f(x)\,dx - \int_{0}^{12} f(x)\,dx$$

$$= 1 - \int_{0}^{12} \frac{x}{9} e^{-x/3}\,dx$$

$$= 1 - \left[-\frac{3x}{9} e^{-x/3} - e^{-x/3} \right]_0^{12}$$

$$= 1 - \left[-4e^{-4} - e^{-4} + 1 \right]$$

$$= 5e^{-4}$$

$$\approx 0.09$$

	D	I
$\oplus$	$x/9$	$e^{-x/3}$
$\ominus$	$1/9$	$-3e^{-x/3}$
		$9e^{-x/3}$

For a C.R.V. X with p.d.f. $f(x)$,
the mean or expected value of X is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

the variance of X is

$$\sigma^2 = E(X^2) - \mu^2 \text{ where } E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

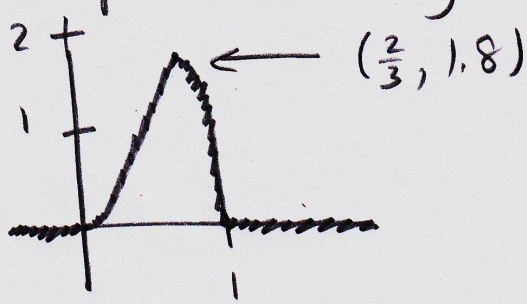
the SD of X is

$$\sigma = \sqrt{\sigma^2}$$

Ex: The proportion of a city's roads needing repair in any given year has p.d.f.

$$f(x) = \begin{cases} 12x^2 - 12x^3, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

a) Graph $f(x)$ using Wolfram Alpha



b) Find the expected proportion of roads needing repair this year

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} (12x^3 - 12x^4)\,dx + \int_{1}^{\infty} 0\,dx$$

$$= \left[3x^4 - \frac{12x^5}{5}\right]_0^1$$

$$= 3 - \frac{12}{5}$$

$$= 0.6$$

c) Find the SD of X

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} (12x^4 - 12x^5)\,dx + \int_{1}^{\infty} 0\,dx$$

$$= \left[\frac{12x^5}{5} - 2x^6\right]_0^1$$

$$= \frac{12}{5} - 2$$

$$= 0.4$$

$$\sigma^2 = E(X^2) - \mu^2$$

$$= 0.4 - 0.6^2$$

$$= 0.04$$

$$\sigma = \sqrt{\sigma^2} = 0.2$$

Rough Check:
If range of X is finite,
$\sigma \approx \frac{\text{range}}{4}$
Here $\frac{\text{range}}{4} = \frac{1-0}{4} = 0.25$

Ex: Find the average daily consumption of electricity from the earlier example

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} \frac{x^2}{9} e^{-x/3}\,dx$$

	D	I
$+$	$x^2/9$	$e^{-x/3}$
$-$	$2x/9$	$-3e^{-x/3}$
$+$	$2/9$	$9e^{-x/3}$
		$-27e^{-x/3}$

$$= \left[\left(\frac{-3x^2}{9} - 2x - \frac{54}{9}\right) e^{-x/3}\right]_0^{\infty}$$

$$\boxed{\begin{array}{l} \text{As } x \to \infty,\ e^{-x} \to 0 \\ \qquad\qquad e^{-x/3} \to 0 \end{array}}$$

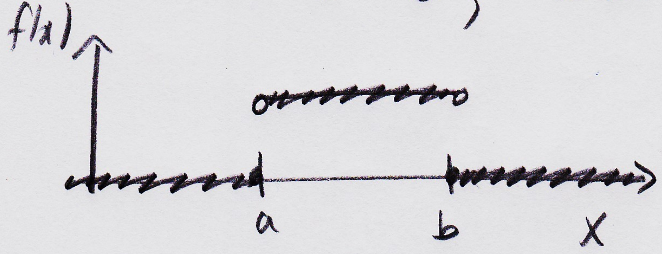
$$= 0 - \left(-\frac{54}{9}\right)e^0$$

$$= 6$$

$$\boxed{6 \text{ million kWh}}$$

A C.R.V. is a **uniform** random variable if its p.d.f. has the form

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$



Ex: Consider a train wheel of radius r. Let x be the location along the circumference (relative to some fixed point o) at which the wheel makes contact with the rail upon braking. X has p.d.f.

$$f(x) = \begin{cases} \frac{1}{2\pi r}, & 0 < x < 2\pi r \\ 0, & \text{otherwise} \end{cases}$$

a) Find the mean of X

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_0^{2\pi r} \frac{x}{2\pi r}\, dx$$

$$= \left[\frac{x^2}{4\pi r} \right]_0^{2\pi r}$$

$$= \frac{(2\pi r)^2}{4\pi r}$$

$$= \pi r$$

b) Find the variance of X

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_0^{2\pi r} \frac{x^2}{2\pi r}\, dx$$

$$= \left[\frac{x^3}{6\pi r} \right]_0^{2\pi r}$$

$$= \frac{8\pi^3 r^3}{6\pi r}$$

$$= \frac{4\pi^2 r^2}{3}$$

$$\sigma^2 = E(x^2) - \mu^2$$

$$= \frac{4\pi^2 r^2}{3} - \pi^2 r^2$$

$$= \frac{\pi^2 r^2}{3}$$

A C.R.V. is an <u>exponential</u> random variable if its p.d.f. has the form

$$f(x) = \begin{cases} ke^{-kx}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

for some constant k.

Shortcut: $\int ke^{-kx}\,dx = -e^{-kx} + C$

<u>Ex</u>: The mileage (in thousands of miles) for a certain kind of tire is a C.R.V. with p.d.f.

$$f(x) = \begin{cases} \frac{1}{20}e^{-x/20}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

Find the probability that a tire lasts:

a) exactly 10,000 miles

$$P(X=10) = 0$$

b) at most 10,000 miles

$$P(X \leq 10) = \int_{-\infty}^{10} f(x)\,dx$$

$$= \int_0^{10} \frac{1}{20} e^{-x/20}\,dx$$

$$= \left[-e^{-x/20}\right]_0^{10}$$

$$= -e^{-0.5} + 1$$

$$\simeq 0.39$$

b) at least 30,000 miles

$$P(X \geq 30) = 1 - P(X < 30)$$

$$= 1 - \int_{-\infty}^{30} f(x)\,dx$$

$$= 1 - \int_0^{30} \frac{1}{20} e^{-x/20}\,dx$$

$$= 1 - \left[-e^{-x/20}\right]_0^{30}$$

$$= 1 - \left[-e^{-1.5} + 1\right]$$

$$= e^{-1.5}$$

$$\simeq 0.22$$

c) between 16,000 and 24,000 miles

$$P(16 \leq X \leq 24) = \int_{16}^{24} f(x)\,dx$$

$$= \int_{16}^{24} \frac{1}{20} e^{-x/20} \, dx$$

$$= \left[-e^{-x/20} \right]_{16}^{24}$$

$$= -e^{-24/20} + e^{-16/20}$$

$$\approx 0.15$$

<u>Ex</u>: If the number of arrivals per unit time is λ, then the waiting time between arrivals is an exponential random variable with $f(x) = \lambda e^{-\lambda x}$ for $x > 0$. Suppose that, on average, 3 trucks arrive at a warehouse per hour. The time (in hours) between truck arrivals has p.d.f.

$$f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the probability that the time between arrivals is:

a) less than 5 minutes

$$5 \text{ minutes} \times \frac{1 \text{ hour}}{60 \text{ minutes}} = \frac{1}{12} \text{ hour}$$

$P(X < \frac{1}{12})$

$= \int_{-\infty}^{1/12} f(x)\,dx$

$= \int_{0}^{1/12} 3e^{-3x}\,dx$

$= \left[-e^{-3x}\right]_{0}^{1/12}$

$= -e^{-0.25} + 1$

≈ 0.22

b) more than 45 minutes

$45 \text{ minutes} \times \frac{1 \text{ hour}}{60 \text{ mins}} = \frac{3}{4} \text{ hour}$

$P(X > \frac{3}{4})$

$= \int_{3/4}^{\infty} f(x)\,dx$

$= \int_{3/4}^{\infty} 3e^{-3x}\,dx$

$= \left[-e^{-3x}\right]_{3/4}^{\infty}$

$\boxed{\text{As } x \to \infty \quad e^{-3x} \to 0}$

$= 0 + e^{-9/4}$

≈ 0.11