5. Binomial and Poisson Distributions

Two useful discrete random variables

I) The Binomial Distribution

X = # successes in a series of independent, identical success/failure trials

n = # trials
p = probability of success on 1 trial
q = " failure "

$$\boxed{P(x \text{ successes}) = {}_nC_x\, p^x q^{n-x}}$$

<u>Ex</u>: Roll a die 13 times. P(at most three 2's or 3's)?

X = # 2's or 3's

BINOMIAL $n=13$ $\quad p = P(2 \text{ or } 3) = \frac{2}{6} = \frac{1}{3}$ $\quad q = 1-p = \frac{2}{3}$

$$P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= 13C0\left(\tfrac{1}{3}\right)^0\left(\tfrac{2}{3}\right)^{13} + 13C1\left(\tfrac{1}{3}\right)^1\left(\tfrac{2}{3}\right)^{12} + 13C2\left(\tfrac{1}{3}\right)^2\left(\tfrac{2}{3}\right)^{11} + 13C3\left(\tfrac{1}{3}\right)^3\left(\tfrac{2}{3}\right)^{10}$$

$$\simeq 0.32$$

<u>Notes</u>: Trials are <u>independent</u> : one roll does not affect other rolls

Trials are <u>identical</u> : p and q are constant from roll to roll

$$P(x \text{ successes}) = \underbrace{nCx}_{\text{choose which trials are successes}}\ p^{x}\ q^{n-x}$$

p^x: P(x successes in a row)

q^{n-x}: P(n−x failures in a row)

Ex: A drilling company is successful on 82% of drilling attempts. P(at least 7 successes in the next 8 attempts)?

$X = \#$ successful attempts

BINOMIAL $n=8$ $p=0.82$ $q=1-p=0.18$

$P(X \geq 7) = P(X=7) + P(X=8)$

$= 8C7\,(0.82)^7(0.18)^1 + 8C8\,(0.82)^8(0.18)^0$

≈ 0.56

Ex: A dart-thrower hits the target 36% of the time. He does not improve with practice. He throws 10 darts. P(he hits the target 2 or 3 times)?

$X = \#$ times he hits the target

BINOMIAL $n=10$ $p=0.36$ $q=1-p=0.64$

$P(X=2 \text{ or } 3) = P(X=2) + P(X=3)$

$= 10C2\,(0.36)^2(0.64)^8 + 10C3\,(0.36)^3(0.64)^7$

≈ 0.41

Ex: A multiple-choice test has 3 questions, each of which has 4 possible answers. A student guesses randomly on each question.

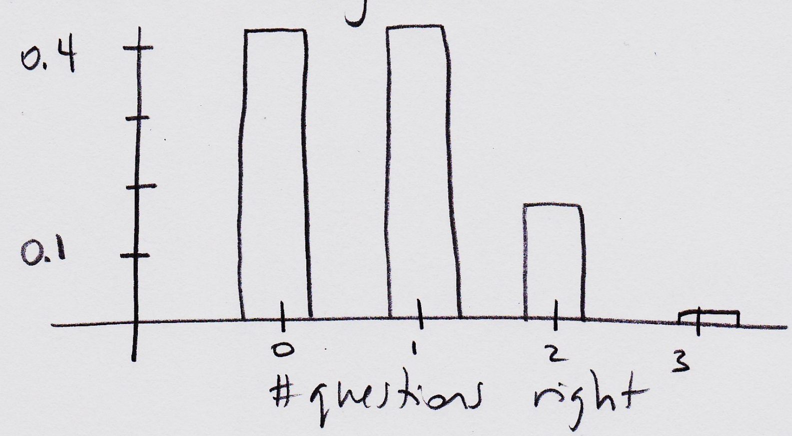
a) Find the probability distribution for # of questions the student gets right.

X = # questions right

BINOMIAL $n=3$ $p=\frac{1}{4}=0.25$ $q=0.75$

x	$P(x)={}_nC_x\, p^x q^{n-x}$
0	${}_3C_0 (0.25)^0(0.75)^3 \approx 0.42$
1	${}_3C_1 (0.25)^1(0.75)^2 \approx 0.42$
2	${}_3C_2 (0.25)^2(0.75)^1 \approx 0.14$
3	${}_3C_3 (0.25)^3(0.75)^0 \approx 0.02$

b) Draw a histogram



II) The Poisson Distribution

X = # occurrences of an event in a unit of time/space

$n! = n \times (n-1) \times \ldots \times 2 \times 1$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$3! = 3 \cdot 2 \cdot 1 = 6$

$2! = 2 \cdot 1 = 2$

$1! = 1$

$0! = 1$ by definition

$$P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

where λ = average # of occurrences in that unit of time/space

Ex: In a city, average # of cracks per m^3 of concrete is 1.9. Find the probability that a randomly-chosen m^3 of concrete has:

a) 2 or 3 cracks

X = # cracks / m^3

Poisson $\lambda = 1.9$

$$P(X=2 \text{ or } 3) = P(X=2) + P(X=3)$$

$$= \frac{1.9^2 e^{-1.9}}{2!} + \frac{1.9^3 e^{-1.9}}{3!}$$

$$\approx 0.44$$

b) at least 3 cracks

$$P(X \geq 3) = 1 - P(X=0) - P(X=1) - P(X=2)$$

$$= 1 - e^{-1.9}\left(\frac{1.9^0}{0!} + \frac{1.9^1}{1!} + \frac{1.9^2}{2!}\right)$$

$$\approx 0.30$$

Ex: A website receives an average of 2 visits per hour.

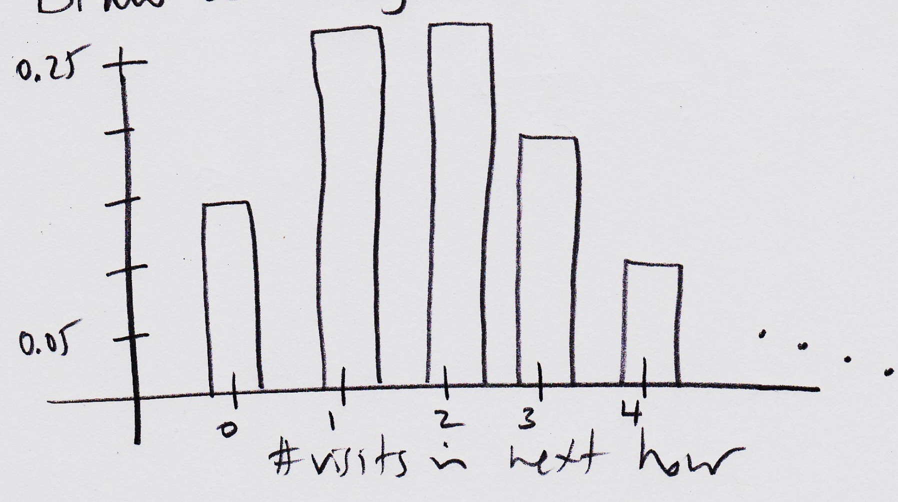
a) Find the probability distribution of # of visits in next hour

X = # visits in next hour

Poisson $\lambda = 2$

x	$P(x) = \frac{2^x e^{-2}}{x!}$
0	0.14
1	0.27
2	0.27
3	0.18
4	0.09
$\vdots$	$\vdots$

b) Draw a histogram



Ex: Concentration of bacteria in the river harbor is 3 per 100mL of water. P(≤2 bacteria in 50mL of water)?

X= # bacteria in $\boxed{50mL}$ of water

Poisson

$$\text{average} = \frac{3}{100mL} = \frac{1.5}{50mL}$$

$$\boxed{\lambda = 1.5}$$

$$P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= e^{-1.5}\left(\frac{1.5^0}{0!} + \frac{1.5^1}{1!} + \frac{1.5^2}{2!}\right)$$

$$\approx 0.81$$

$\boxed{\text{Units of } \lambda \text{ must match units in the question}}$

Ex: There are an average of 1.8 accidents per week on a highway. $P(\geq 4$ accidents in next 2 weeks$)$?

$X =$ # accidents in next 2 weeks

Poisson

$$\text{average} = \frac{1.8 \text{ accidents}}{\text{week}} = \frac{3.6 \text{ accidents}}{2 \text{ weeks}}$$

$$\boxed{\lambda = 3.6}$$

$$P(X \geq 4) = 1 - P(X=0) - P(X=1) - P(X=2) - P(X=3)$$

$$= 1 - e^{-3.6}\left(\frac{3.6^0}{0!} + \frac{3.6^1}{1!} + \frac{3.6^2}{2!} + \frac{3.6^3}{3!}\right)$$

$$\approx 0.48$$

BINOMIAL: $X =$ #successes in $\boxed{\text{repeated trials}}$

e.g. rolling a die
shooting at a target

POISSON: $X =$ # occurrences in a unit of $\boxed{\text{time/space}}$

e.g. #cracks / m^3
#website visits /hr