31.9 · Solutions of Nonhomogeneous DE

Ex: Solve $D^2y - 8Dy + 15y = 7e^{2x}$

$\boxed{\begin{array}{c} f(x) \neq 0 \\ \text{DE is nonhomogeneous} \end{array}}$

1) Find y_c, the "complementary solution"

Set $D^2y - 8Dy + 15y = 0$

$$m^2 - 8m + 15 = 0$$
$$(m-3)(m-5) = 0$$
$$m = 3, 5$$
$$y_c = C_1 e^{3x} + C_2 e^{5x}$$

2) Set up y_p, the "particular solution"

$\boxed{\text{$y_p$ has unknown coefficients and involves all terms in } f(x), f'(x), f''(x), \ldots}$

$$\left.\begin{array}{l} f(x) = 7e^{2x} \\ f'(x) = 14e^{2x} \\ f''(x) = 28e^{2x} \end{array}\right\} \text{like terms}$$

etc.

$$y_p = Ae^{2x}$$

3) $y_p \to DE$ (find A)

$$\left.\begin{aligned} y_p &= Ae^{2x} \\ y_p' &= 2Ae^{2x} \\ y_p'' &= 4Ae^{2x} \end{aligned}\right\} \to DE$$

$$y'' - 8y' + 15y = 7e^{2x}$$

$$4Ae^{2x} - 8(2Ae^{2x}) + 15(Ae^{2x}) = 7e^{2x}$$

$$(4A - 16A + 15A)e^{2x} = 7e^{2x}$$

$$3Ae^{2x} = 7e^{2x}$$

match coefficients

$$3A = 7$$

$$A = \frac{7}{3}$$

Conclude $y_p = \frac{7}{3}e^{2x}$

4) $y = y_c + y_p$

$$\boxed{y = C_1e^{3x} + C_2e^{5x} + \frac{7}{3}e^{2x}}$$

Setting up y_p:

If $f(x) = 5x^2$

$f'(x) = 10x$

$f''(x) = 10$

$f'''(x) = 0 \leftarrow$ stop when no new like terms appear

$y_p = Ax^2 + Bx + C$

$f(x)$	y_p
$7e^{2x}$	Ae^{2x}
$6x^2 - 4x$	$Ax^2 + Bx + C$
$9e^{6x} - 3x$	$Ae^{6x} + Bx + C$
$8\sin 2x$	$A\sin 2x + B\cos 2x$
$8\sin 2x - 4\cos 3x$	$A\sin 2x + B\cos 2x + C\sin 3x + D\cos 3x$
$x\sin x$	$Ax\sin x + Bx\cos x + C\sin x + D\cos x$
$xe^{4x} + 1$	$Axe^{4x} + Be^{4x} + C$

$f(x) = x\sin x$

$f'(x) = x\cos x + \sin x$

$f''(x) = -x\sin x + \cos x + \cos x$

$f'''(x) = -x\cos x - \sin x - \sin x - \sin x$

No new terms

$f(x) = xe^{4x} + 1$

$f'(x) = x(4e^{4x}) + e^{4x}$

$f''(x) = x(16e^{4x}) + 4e^{4x} + 4e^{4x}$

No new terms

Ex: Solve $D^2y + 9y = 2\sin x$

1) y_c

$$y'' + 9y = 0$$

$$m^2 + 9 = 0$$

$$m = \frac{\pm\sqrt{-36}}{2}$$

$$m = \frac{\pm 6j}{2}$$

$$m = \pm 3j \quad (\alpha = 0, \beta = 3)$$

$$y_c = e^{0x}(C_1 \sin 3x + C_2 \cos 3x)$$

$$\boxed{y_c = C_1 \sin 3x + C_2 \cos 3x}$$

2) y_p

$f(x) = 2\sin x$

$y_p = A\sin x + B\cos x$

3) $y_p \to DE$

$$\left.\begin{aligned} y_p &= A\sin x + B\cos x \\ y_p' &= A\cos x - B\sin x \\ y_p'' &= -A\sin x - B\cos x \end{aligned}\right\} \to DE$$

$$y'' + 9y = 2\sin x$$

$$-A\sin x - B\cos x + 9(A\sin x + B\cos x) = 2\sin x$$

$$8A\sin x + 8B\cos x = 2\sin x + 0\cos x$$

$$\boxed{\begin{aligned} 8A &= 2 \\ A &= \tfrac{1}{4} \end{aligned}} \qquad \boxed{\begin{aligned} 8B &= 0 \\ B &= 0 \end{aligned}}$$

$$y_p = \frac{1}{4}\sin x$$

4) $y = y_c + y_p$

$$\boxed{y = C_1\sin 3x + C_2\cos 3x + \frac{1}{4}\sin x}$$

The Bad Case

When y_p and y_c have like terms, multiply the relevant terms of y_p by the smallest power of x that avoids like terms.

Ex: Solve $D^2y - 4Dy + 4y = 12 + 2e^{2x}$

1) y_c

$$y'' - 4y' + 4y = 0$$

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$m = 2, 2$ repeated roots

$$y_c = (C_1 + C_2 x)e^{2x}$$

2) y_p

$y_p = A + Be^{2x}$ won't work

Bad Case: e^{2x} and xe^{2x} appear in y_c

$$y_p = A + Bx^2e^{2x}$$

ok

Multiply by x^2 to avoid like terms with y_c

3) $y_p \to DE$

$$\boxed{y_p = A + Bx^2 e^{2x}}$$

$$y_p' = Bx^2(2e^{2x}) + 2Bxe^{2x}$$

$$\boxed{y_p' = (2Bx^2 + 2Bx)e^{2x}}$$

$$y_p'' = (2Bx^2 + 2Bx)(2e^{2x}) + e^{2x}(4Bx + 2B)$$

$$\boxed{y_p'' = (4Bx^2 + 8Bx + 2B)e^{2x}}$$

$$y'' - 4y' + 4y = 12 + 2e^{2x}$$

$$(4Bx^2 + 8Bx + 2B)e^{2x} - 4(2Bx^2 + 2Bx)e^{2x}$$

$$+ 4\left[A + Bx^2 e^{2x}\right] = 12 + 2e^{2x}$$

$$0x^2e^{2x} + 0xe^{2x} + 2Be^{2x} + 4A = 12 + 2e^{2x}$$

$$\boxed{\begin{array}{c} 2B = 2 \\ B = 1 \end{array}} \quad \boxed{\begin{array}{c} 4A = 12 \\ A = 3 \end{array}}$$

$$y_p = 3 + x^2 e^{2x}$$

4) $y = y_c + y_p$

$$\boxed{y = (C_1 + C_2 x)e^{2x} + 3 + x^2 e^{2x}}$$

Ex: DE has $y_c = C_1 \sin x + C_2 \cos x$

Write down y_p

a) $y'' + y = e^{5x} + x$

$y_p = Ae^{5x} + Bx + C$ (not bad case)

b) $y'' + y = 2\cos 7x$

$y_p = A\sin 7x + B\cos 7x$ (not bad case)

c) $y'' + y = 3\sin x + 7$

$y_p = A\sin x + B\cos x + C$ won't work

$y_p = Ax\sin x + Bx\cos x + C$

OK

<u>Ex</u>: Solve $y'' - 8y' + 16y = x^2$

1) y_c

$$m^2 - 8m + 16 = 0$$
$$(m-4)^2 = 0$$
$$m = 4, 4$$
$$y_c = (C_1 + C_2 x)e^{4x}$$

2) $y_p = Ax^2 + Bx + C$ (not bad case)

3) $y_p \to DE$

$$\left.\begin{aligned} y_p &= Ax^2 + Bx + C \\ y_p' &= 2Ax + B \\ y_p'' &= 2A \end{aligned}\right\} \to DE$$

$$y'' - 8y' + 16y = x^2$$

$$2A - 8(2Ax + B) + 16(Ax^2 + Bx + C) = x^2$$

$$16Ax^2 + (16B - 16A)x + (2A - 8B + 16C) = x^2 + 0x + 0$$

$$\boxed{\begin{aligned} 16A &= 1 \\ A &= \tfrac{1}{16} \end{aligned}} \quad \boxed{\begin{aligned} 16B - 16A &= 0 \\ 16B - 1 &= 0 \\ B &= \tfrac{1}{16} \end{aligned}} \quad \boxed{\begin{aligned} 2A - 8B + 16C &= 0 \\ \tfrac{2}{16} - \tfrac{8}{16} + 16C &= 0 \\ 16C &= \tfrac{6}{16} \\ C = \tfrac{6}{256} &= \tfrac{3}{128} \end{aligned}}$$

$$y_p = \frac{1}{16}x^2 + \frac{1}{16}x + \frac{3}{128}$$

4) $y = y_c + y_p$

$$\boxed{y = (C_1 + C_2 x)e^{4x} + \frac{x^2}{16} + \frac{x}{16} + \frac{3}{128}}$$

<u>Ex</u>: Solve $y'' + 25y = 4e^{-3x}$
if $y' = 2$ and $y = 1$ when $x = 0$

1) y_c

$$m^2 + 25 = 0$$

$$m = \frac{\pm\sqrt{-100}}{2}$$

$$m = \frac{\pm 10j}{2}$$

$$m = \pm 5j \quad (\alpha = 0, \beta = 5)$$

$$y_c = e^{0x}(C_1 \sin 5x + C_2 \cos 5x)$$

$$\boxed{y_c = C_1 \sin 5x + C_2 \cos 5x}$$

2) y_p

$y_p = Ae^{-3x}$ (not bad case)

3) $y_p \to DE$

$$\left.\begin{aligned} y_p &= Ae^{-3x} \\ y_p' &= -3Ae^{-3x} \\ y_p'' &= 9Ae^{-3x} \end{aligned}\right\} \to DE$$

$$y'' + 25y = 4e^{-3x}$$

$$9Ae^{-3x} + 25Ae^{-3x} = 4e^{-3x}$$

$$34Ae^{-3x} = 4e^{-3x}$$

$$\boxed{\begin{aligned} 34A &= 4 \\ A &= \frac{4}{34} = \frac{2}{17} \end{aligned}}$$

$$y_p = \frac{2}{17}e^{-3x}$$

4) $y = y_c + y_p$

$$\boxed{y = C_1 \sin 5x + C_2 \cos 5x + \frac{2}{17}e^{-3x}}$$

5) Use initial conditions to find C_1, C_2

$\begin{matrix} y=1 \\ x=0 \end{matrix}$: $1 = C_2 + \frac{2}{17}$

$$\boxed{C_2 = \frac{15}{17}}$$

$$\boxed{y = C_1 \sin 5x + \frac{15}{17} \cos 5x + \frac{2}{17} e^{-3x}}$$

$$y' = 5C_1 \cos 5x - \frac{75}{17} \sin 5x - \frac{6}{17} e^{-3x}$$

$\begin{matrix} y' = 2 \\ x = 0 \end{matrix}$:

$$2 = 5C_1 - \frac{6}{17}$$

$$\frac{34}{17} + \frac{6}{17} = 5C_1$$

$$\frac{40}{17} = 5C_1$$

$$\boxed{C_1 = \frac{8}{17}}$$

$$\boxed{y = \frac{8}{17} \sin 5x + \frac{15}{17} \cos 5x + \frac{2}{17} e^{-3x}}$$