31.8 Auxiliary Equation with Repeated or Complex Roots

Repeated Roots

If root m is repeated twice
then $y = (C_1 + C_2 x)e^{mx}$

If root m is repeated 3 times
then $y = (C_1 + C_2 x + C_3 x^2)e^{mx}$
etc.

Reminder on formula sheet $y = (C_1 + C_2 x)e^{mx}$

Ex: Solve $D^2 y + 4Dy + 4y = 0$

$$m^2 + 4m + 4 = 0$$

$$(m+2)^2 = 0$$

$$m = -2, -2 \quad \text{repeated roots}$$

$$y = (C_1 + C_2 x)e^{-2x}$$

Quick Ex:

a) $m = 6, 6, 6$

$$y = (C_1 + C_2 x + C_3 x^2)e^{6x}$$

b) $m = 6, 7, 7$

$$y = C_1 e^{6x} + (C_2 + C_3 x)e^{7x}$$

c) $m = -2, -2, 3, 3$

$$y = (C_1 + C_2 x)e^{-2x} + (C_3 + C_4 x)e^{3x}$$

Ex: Solve $D^5 y - 6D^4 y + 9D^3 y = 0$

$$m^5 - 6m^4 + 9m^3 = 0$$

$$m^3(m^2 - 6m + 9) = 0$$

$$m^3(m-3)^2 = 0$$

$$m = 0, 0, 0, 3, 3$$

$$y = (C_1 + C_2 x + C_3 x^2)e^{0x} + (C_4 + C_5 x)e^{3x}$$

$$y = C_1 + C_2 x + C_3 x^2 + (C_4 + C_5 x)e^{3x}$$

Complex Numbers

$j = \sqrt{-1}$ (sometimes called i)

$\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2j$

$\sqrt{-7} = \sqrt{7}\sqrt{-1} = \sqrt{7}j$ or $j\sqrt{7}$

Complex Roots

If roots $m = \alpha \pm \beta j$ then

$$y = e^{\alpha x}(C_1 \sin\beta x + C_2 \cos\beta x)$$

Note: no $\pm$, no j in solution

Ex: Solve $D^2y + 2Dy + 2y = 0$

$$m^2 + 2m + 2 = 0$$

$$m = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2}$$

$$m = \frac{-2 \pm \sqrt{-4}}{2} \leftarrow \sqrt{4}\sqrt{-1}$$

$$m = \frac{-2 \pm 2j}{2}$$

$$m = -1 \pm j$$

$\boxed{\begin{array}{l}\alpha = -1 \\ \beta = 1\end{array}}$ $y = e^{-x}(C_1 \sin x + C_2 \cos x)$

Ex: Solve $D^3y + 25Dy = 0$

$$m^3 + 25m = 0$$

$$m(m^2+25) = 0$$

$m = 0$ $\quad$ $m = \dfrac{\pm\sqrt{-100}}{2}$

$$m = \frac{\pm 10j}{2}$$

$$m = \pm 5j \qquad (\alpha = 0, \beta = 5)$$

$$y = e^{0x}(C_1 \sin 5x + C_2 \cos 5x) + C_3 e^{0x}$$

$$\boxed{y = C_1 \sin 5x + C_2 \cos 5x + C_3}$$

Ex: Solve $D^2y + Dy = -y$

$$D^2y + Dy + y = 0$$

$$m^2 + m + 1 = 0$$

$$m = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$m = \frac{-1 \pm \sqrt{-3}}{2}$$

$$m = \frac{-1 \pm \sqrt{3}\,j}{2}$$

$$m = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} j \quad \left(\alpha = \frac{-1}{2} \quad \beta = \frac{\sqrt{3}}{2}\right)$$

$$y = e^{-x/2}\left(C_1 \sin \frac{\sqrt{3}}{2} x + C_2 \cos \frac{\sqrt{3}}{2} x\right)$$

Ex: Solve $y'' - 2y' + 8y = 0$ if $y = 1$ and $y' = 6$ when $x = 0$

$$m^2 - 2m + 8 = 0$$

$$m = \frac{2 \pm \sqrt{4-32}}{2}$$

$$m = \frac{2 \pm \sqrt{-28}}{2} \leftarrow \sqrt{4}\sqrt{7}\sqrt{-1}$$

$$m = \frac{2 \pm 2\sqrt{7}\,j}{2}$$

$$m = 1 \pm \sqrt{7}\,j \quad (\alpha = 1 \quad \beta = \sqrt{7})$$

$$y = e^{x}\left(C_1 \sin\sqrt{7}x + C_2 \cos\sqrt{7}x\right)$$

$y=1$, $x=0$: $\quad 1 = 1(0 + C_2)$

$$1 = C_2$$

$\sin 0 = 0$
$\cos 0 = 1$

$$y = e^{x}\left(C_1 \sin\sqrt{7}x + \cos\sqrt{7}x\right)$$

$$y' = e^{x}\left(\sqrt{7}C_1 \cos\sqrt{7}x - \sqrt{7}\sin\sqrt{7}x\right)$$
$$+ \left(C_1 \sin\sqrt{7}x + \cos\sqrt{7}x\right)e^{x}$$ Product Rule

$y'=6$, $x=0$: $\quad 6 = 1(\sqrt{7}C_1 - 0) + (0+1)1$

$$6 = \sqrt{7}C_1 + 1$$

$$5 = \sqrt{7}C_1$$

$$C_1 = \frac{5}{\sqrt{7}} = \frac{5\sqrt{7}}{7}$$

$$y = e^{x}\left(\frac{5\sqrt{7}}{7}\sin\sqrt{7}x + \cos\sqrt{7}x\right)$$

Ex: Solve $D^4y - 16y = 0$

$$m^4 - 16 = 0$$

Sub $a = m^2$: $a^2 - 16 = 0$

$$(a-4)(a+4) = 0$$

$$(m^2-4)(m^2+4) = 0$$

$$(m-2)(m+2)(m^2+4) = 0$$

$m = 2$ $\quad m = -2 \quad$ $m = \frac{\pm\sqrt{-16}}{2}$

$$m = \frac{\pm 4j}{2}$$

$$m = \pm 2j$$

$$(\alpha = 0, \beta = 2)$$

$$y = e^{0x}(C_1 \sin 2x + C_2 \cos 2x) + C_3 e^{2x} + C_4 e^{-2x}$$

$$y = C_1 \sin 2x + C_2 \cos 2x + C_3 e^{2x} + C_4 e^{-2x}$$

Repeated Complex Roots

If $m = \alpha \pm \beta j,\ \alpha \pm \beta j$

$$y = e^{\alpha x}(C_1 \sin \beta x + C_2 \cos \beta x)$$

$$+ e^{\alpha x}(C_3\, x \sin \beta x + C_4\, x \cos \beta x)$$

↑ ↑ due to repeated root

Ex: Solve $D^4 y + 8D^2 y + 16y = 0$

$$m^4 + 8m^2 + 16 = 0$$

$a = m^2$: $\quad a^2 + 8a + 16 = 0$

$$(a+4)(a+4) = 0$$

$$(m^2+4)(m^2+4) = 0$$

$m^2 + 4 = 0 \qquad m = \pm 2j$

$$m = \frac{\pm\sqrt{-16}}{2}$$

$$m = \frac{\pm 4j}{2}$$

$m = \pm 2j \qquad (\alpha = 0, \beta = 2)$

$$y = e^{0x}(C_1 \sin 2x + C_2 \cos 2x)$$
$$+ e^{0x}(C_3 \underset{\uparrow}{x} \sin 2x + C_4 \underset{\uparrow}{x} \cos 2x)$$

or $y = (C_1 + C_3 x)\sin 2x + (C_2 + C_4 x)\cos 2x$

Recap

Roots	Solution
4, 4, 4	$y = (C_1 + C_2 x + C_3 x^2)e^{4x}$
1, 5, 5	$y = C_1 e^{x} + (C_2 + C_3 x)e^{5x}$
$2 \pm 3j$	$y = e^{2x}(C_1 \sin 3x + C_2 \cos 3x)$
$1, 2 \pm 3j$	$y = C_1 e^{x} + e^{2x}(C_2 \sin 3x + C_3 \cos 3x)$
$2 \pm 3j, 2 \pm 3j$	$y = e^{2x}(C_1 \sin 3x + C_2 \cos 3x) + e^{2x}(C_3 \underset{\uparrow}{x} \sin 3x + C_4 \underset{\uparrow}{x} \cos 3x)$

or $y = e^{2x}\left[(C_1 + C_3 x)\sin 3x + (C_2 + C_4 x)\cos 3x\right]$