31.7 Higher-Order Homogeneous DE with Constant Coefficients

Notation:
$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + y = f(x)$$

or
$$y''' - 4y'' + 5y' + y = f(x)$$

or
$$D^3y - 4D^2y + 5Dy + y = f(x)$$

The DE is <u>homogeneous</u> if $f(x) = 0$

<u>non-homogeneous</u> if $f(x) \neq 0$

<u>Ex</u>: Rewrite $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = e^x$

$$y'' + 5y' = e^x$$

or
$$D^2y + 5Dy = e^x$$

DE is non-homogeneous

Ex: Solve $y'' - 7y' + 12y = 0$

"auxiliary equation" $m^2 - 7m + 12 = 0$

$(m-3)(m-4) = 0$

$m = 3, 4$

$$\boxed{\text{General Solution is } y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}}$$

$$\boxed{y = C_1 e^{3x} + C_2 e^{4x}}$$

Check: $y = C_1 e^{3x} + C_2 e^{4x}$

$y' = 3C_1 e^{3x} + 4C_2 e^{4x}$

$y'' = 9C_1 e^{3x} + 16C_2 e^{4x}$

$$LS = y'' - 7y' + 12y$$

$$= 9C_1 e^{3x} + 16C_2 e^{4x} - 7(3C_1 e^{3x} + 4C_2 e^{4x})$$

$$+ 12(C_1 e^{3x} + C_2 e^{4x})$$

$$= 0C_1 e^{3x} + 0C_2 e^{4x}$$

$$= 0$$

$$= RS \checkmark$$

Auxiliary equation guarantees that $LS = 0$

$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}$ when:

DE is homogeneous

and Roots of auxiliary equation are <u>real</u> and <u>distinct</u>

Section 31.8 Repeated or Complex Roots

31.9 Non-homogeneous DE

<u>Ex</u>: Solve $D^3 y = 4Dy$

$$y''' - 4y' = 0$$

$$m^3 - 4m = 0$$

$$m(m^2 - 4) = 0$$

$$m(m-2)(m+2) = 0$$

$$m = 0, 2, -2$$

$$y = C_1 e^{0x} + C_2 e^{2x} + C_3 e^{-2x}$$

$$\boxed{y = C_1 + C_2 e^{2x} + C_3 e^{-2x}}$$

Ex: Solve $y'' - 8y' = -5y$

$$y'' - 8y' + 5y = 0$$

$$m^2 - 8m + 5 = 0$$

$$m = \frac{8 \pm \sqrt{(-8)^2 - 4(1)(5)}}{2}$$

$$m = \frac{8 \pm \sqrt{44}}{2} \leftarrow \sqrt{4}\sqrt{11}$$

$$m = \frac{8 \pm 2\sqrt{11}}{2}$$

$$m = 4 \pm \sqrt{11}$$

$$y = C_1 e^{(4+\sqrt{11})x} + C_2 e^{(4-\sqrt{11})x}$$

$$y = C_1 e^{4x} e^{\sqrt{11}x} + C_2 e^{4x} e^{-\sqrt{11}x}$$

$$y = e^{4x} \left(C_1 e^{\sqrt{11}x} + C_2 e^{-\sqrt{11}x} \right)$$

Ex: Solve $D^2y - 5Dy - 14y = 0$
given $y = 11$ and $Dy = 5$ when $x = 0$

$$m^2 - 5m - 14 = 0$$
$$(m-7)(m+2) = 0$$
$$m = 7, -2$$
$$\boxed{y = C_1 e^{7x} + C_2 e^{-2x}} \quad ★$$

$y=11$, $x=0$: $\quad 11 = C_1 + C_2 \quad (1)$

$$Dy = 7C_1 e^{7x} - 2C_2 e^{-2x}$$

$Dy=5$, $x=0$: $\quad 5 = 7C_1 - 2C_2 \quad (2)$

$2 \times (1)$: $\quad 22 = 2C_1 + 2C_2$
$+ \quad (2)$: $\quad 5 = 7C_1 - 2C_2$

$$27 = 9C_1$$
$$\boxed{C_1 = 3}$$

$C_1 = 3 \to (1)$: $\quad 11 = 3 + C_2$
$$\boxed{C_2 = 8}$$

$$★ \quad \boxed{y = 3e^{7x} + 8e^{-2x}}$$

Ex: Solve $3y'' + 5y' + 2y = 0$

$$3m^2 + 5m + 2 = 0$$

$$m = \frac{-5 \pm \sqrt{25 - 4(3)(2)}}{6}$$

$$m = \frac{-5 \pm \sqrt{1}}{6}$$

$$m = \frac{-6}{6}, \frac{-4}{6}$$

$$m = -1, -\frac{2}{3}$$

$$y = C_1 e^{-x} + C_2 e^{\frac{-2x}{3}}$$

Ex: Solve $D^4 y - 10D^2 y + 9y = 0$

$$m^4 - 10m^2 + 9 = 0$$

Let $a = m^2$: $a^2 - 10a + 9 = 0$

$$(a-1)(a-9) = 0$$

$$(m^2-1)(m^2-9) = 0$$

$$(m-1)(m+1)(m-3)(m+3) = 0$$

$$m = \pm 1, \pm 3$$

$$y = C_1 e^{x} + C_2 e^{-x} + C_3 e^{3x} + C_4 e^{-3x}$$

<u>Ex</u>: Solve $y'' - y' - 20y = 0$
if $y = 0$ when $x = 0$
and $y = 1$ when $x = 1$

$$m^2 - m - 20 = 0$$
$$(m-5)(m+4) = 0$$
$$m = 5, -4$$
$$\boxed{y = C_1 e^{5x} + C_2 e^{-4x}} \quad \text{A}$$

$y=0$, $x=0$: $\quad 0 = C_1 + C_2 \quad$ 1

$y=1$, $x=1$: $\quad 1 = C_1 e^5 + C_2 e^{-4} \quad$ 2

$-e^4 \times$ 2 $\quad -e^4 = -C_1 e^9 - C_2$

1 $\quad 0 = C_1 + C_2$

$+$

$$-e^4 = C_1 - C_1 e^9$$
$$-e^4 = C_1(1 - e^9)$$
$$\boxed{C_1 = \frac{-e^4}{1-e^9} = \frac{e^4}{e^9 - 1}}$$

$C_1 = \dfrac{e^4}{e^9-1} \rightarrow (1): \quad 0 = \dfrac{e^4}{e^9-1} + C_2$

$$\boxed{C_2 = -\dfrac{e^4}{e^9-1}}$$

(A) $y = \dfrac{e^4}{e^9-1} e^{5x} - \dfrac{e^4}{e^9-1} e^{-4x}$ ✓

$y = \dfrac{e^4}{e^9-1}\left(e^{5x} - e^{-4x}\right)$ ✓