31.6 Elementary Applications

Ex: Find the explicit equation of the curve with slope $= y+2x$ passing through $(0,3)$

$$\frac{dy}{dx} = y+2x \qquad \boxed{\text{Linear}}$$

1) Standard

$$dy = y\,dx + 2x\,dx$$

$$\boxed{dy - y\,dx = 2x\,dx}$$

2) I.F.

$$P(x) = -1$$

$$\int P(x)\,dx = -x$$

$$\text{I.F.} = e^{-x}$$

3) $e^{-x}dy - e^{-x}y\,dx = 2xe^{-x}dx$

$$d(e^{-x}y) = 2xe^{-x}dx$$

4) $\int d(e^{-x}y) = \int 2xe^{-x}dx$

	D	I
$\oplus$	$2x$	e^{-x}
$\ominus$	2	$-e^{-x}$
		e^{-x}

$$e^{-x}y = -2xe^{-x} - 2e^{-x} + C$$

$$\boxed{y = -2x - 2 + Ce^{x}}$$

$x=0$, $y=3$: $3 = -2 + C$

$C = 5 \rightarrow \boxed{y = -2x - 2 + 5e^{x}}$

Exponential Growth and Decay

e.g. population growth
radioactive decay

Physical Principle:
Rate of growth/decay is proportional to quantity present

Let N = quantity
t = time

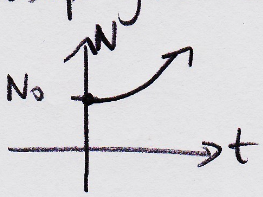
$$\frac{dN}{dt} \propto N$$

$$\boxed{\frac{dN}{dt} = kN}$$ DE for Exp. Growth/Decay
k: constant

$\vdots$

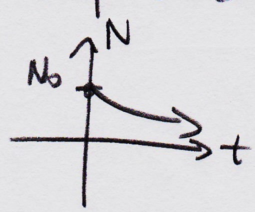
$N = N_0 e^{kt}$ $\quad N_0$ = initial quantity

$k > 0$
exp. growth



$N \to \infty$ as $t \to \infty$

$k < 0$
exp. decay



$N \to 0$ as $t \to \infty$

Ex: A radioactive substance has a half-life of 7.9 years. How long for 90% of substance to decay?

$\boxed{\text{half-life} = \text{time for } 50\% \text{ of substance to decay}}$

Let N = quantity
N_0 = initial quantity
t = time (years)

$$\boxed{\frac{dN}{dt} = kN}$$

Separable

$$\frac{dN}{N} = k\,dt$$

$$\int \frac{dN}{N} = \int k\,dt$$

$$\ln N = kt + C_1$$

$$e^{LS} = e^{RS}: \quad e^{\ln N} = \cancel{e^{kt+C_1}} \leftarrow e^{kt}\cdot e^{C_1} = C$$

$$\boxed{N = Ce^{kt}}$$

$N = N_0$, $t = 0$: $\quad N_0 = C$

$$\boxed{N = N_0 e^{kt}}$$

$N = 0.5 N_0$; $\quad 0.5 N_0 = N_0 e^{k(7.9)}$

$t = 7.9$

$$0.5 = e^{7.9k}$$

$$\ln 0.5 = 7.9k$$

$$k = \frac{\ln 0.5}{7.9}$$

$$\boxed{N = N_0 e^{\frac{\ln 0.5}{7.9} t}}$$

90% decayed
= 10% remaining
$N = 0.1 N_0$:

$$0.1\ N_0 = N_0 e^{kt}$$

$$0.1 = e^{kt}$$

$$\ln 0.1 = kt$$

$$\frac{\ln 0.1}{k} = t$$

$$t = \frac{\ln 0.1}{\left(\frac{\ln 0.5}{7.9}\right)}$$

$$\approx 26 \text{ years}$$

Ex: A population grows exponentially
Population is 5.0 million in 2010
and 8.0 " 2012
Population in 2020?

$t =$ # years after 2010
$N =$ population (millions)

$$\boxed{\frac{dN}{dt} = kN}$$

$$\frac{dN}{N} = k\,dt$$

$$\ln N = kt + C_1$$

$$e^{\ln N} = \cancel{e^{kt+C_1}} \leftarrow e^{kt} e^{C_1} = C$$

$$\boxed{N = Ce^{kt}}$$

$N=5$, $t=0$: $5 = C$

$$\boxed{N = 5e^{kt}}$$

$N=8$, $t=2$: $8 = 5e^{k(2)}$

$$\frac{8}{5} = e^{2k}$$

$$\ln\left(\frac{8}{5}\right) = 2k$$

$k = \frac{1}{2}\ln\left(\frac{8}{5}\right)$

$$\boxed{N = 5e^{\frac{1}{2}\ln\left(\frac{8}{5}\right)t}}$$

$t = 10$: $N = 5e^{5\ln\left(\frac{8}{5}\right)}$
≈ 52

Population in 2020 is approx. 52 million

Newton's Law of Cooling

> Physical Principle: Rate at which an object cools/warms is proportional to temprature difference between it and its environment

Ex: In a 25°C room,

Coffee goes 90°C → 89°C faster than 67°C → 66°C

Cold water goes 3°C → 7°C faster than 8°C → 12°C

T = object's temp. VARIABLE

T_e = environment temp. CONSTANT

t = time

$$\boxed{\frac{dT}{dt} = k(T - T_e) \quad \text{DE for heating/cooling} \quad k: \text{constant}}$$

<u>Ex</u>: How long does it take a cup of coffee, initially 75°C, to cool to 40°C if it takes 6 minutes to cool to 60°C? Room temp is 20°C.

T = coffee temp. (°C)
t = time (mins)

$$\frac{dT}{dt} = k(T-20) \quad \boxed{\text{Separable}}$$

$$\frac{dT}{T-20} = k\,dt$$

$$\ln(T-20) = kt + C_1$$

$$e^{\ln(T-20)} = e^{kt+C_1} \Leftarrow e^{kt} e^{C_1} = C$$

$$T - 20 = Ce^{kt}$$

$$\boxed{T = 20 + Ce^{kt}}$$

$t=0$, $T=75$: $75 = 20 + C$

$C = 55$

$$\boxed{T = 20 + 55e^{kt}}$$

$T=60$, $t=6$: $60 = 20 + 55e^{k(6)}$

$40 = 55e^{6k}$

$\frac{40}{55} = e^{6k}$

$\ln\left(\frac{40}{55}\right) = 6k$

$k = \frac{1}{6}\ln\left(\frac{40}{55}\right)$

$$\boxed{T = 20 + 55e^{\frac{1}{6}\ln\left(\frac{40}{55}\right)t}}$$

$T=40$: $40 = 20 + 55e^{kt}$

$20 = 55e^{kt}$

$\frac{20}{55} = e^{kt}$

$\ln\left(\frac{20}{55}\right) = kt$

$$t = \frac{\ln\left(\frac{20}{55}\right)}{k} = \frac{\ln\left(\frac{20}{55}\right)}{\left[\frac{1}{6}\ln\left(\frac{40}{55}\right)\right]} \simeq 19 \text{ mins}$$

As $t \to \infty$, what happens to T?

$$\boxed{T = 20 + 55e^{kt} \qquad (k<0)}$$

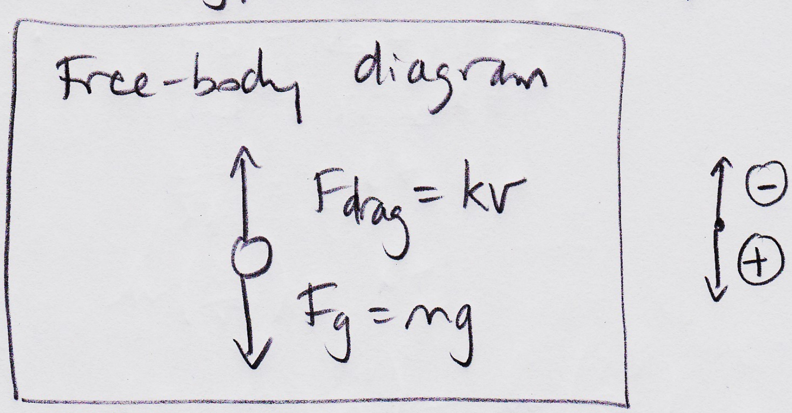
As $t \to \infty$, $\quad e^{kt} \to 0$

$\qquad T \to 20$

Coffee temp approaches room temp.

Ex: An object starts at rest and falls under the influence of gravity, with air resistance proportional to speed. Find a formula for speed v.

Let v = speed
$\quad t$ = time
$\quad m, g, k$ constants (>0)



$$F_{net} = mg - kv$$

$$ma = mg - kv$$

$$\boxed{m\frac{dv}{dt} = mg - kv}$$ Liner DE

1) Standard Form

$$m\,dv = mg\,dt - kv\,dt$$

$$dv = g\,dt - \frac{k}{m}v\,dt$$

$$\boxed{dv} + \frac{k}{m}\boxed{v}dt = g\,dt$$

2) I.F.

$$P(t) = \frac{k}{m}$$

$$\int P(t)\,dt = \frac{k}{m}t$$

$$I.F. = e^{\frac{kt}{m}}$$

3) $$e^{\frac{kt}{m}}dv + \frac{k}{m}e^{\frac{kt}{m}}v\,dt = ge^{\frac{kt}{m}}dt$$

$$d\left(e^{\frac{kt}{m}}v\right) = ge^{\frac{kt}{m}}dt$$

4) Integrate

$$e^{\frac{kt}{m}}v = \frac{mg}{k}e^{\frac{kt}{m}} + C$$

Mult. by $e^{-\frac{kt}{m}}$:

$$\boxed{v = \frac{mg}{k} + Ce^{-\frac{kt}{m}}}$$

$v=0$
$t=0$:

$$0 = \frac{mg}{k} + C$$

$$C = -\frac{mg}{k}$$

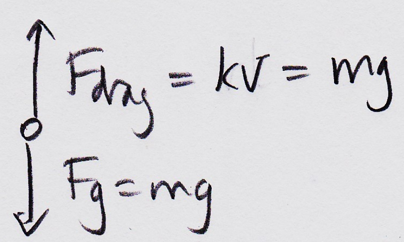
$$\boxed{v = \frac{mg}{k}\left(1 - e^{-\frac{kt}{m}}\right)}$$

As $t \to \infty$, what happens to v?

$$e^{-\frac{kt}{m}} \to 0$$

$$v \to \frac{mg}{k}$$

$v = \frac{mg}{k}$ is "terminal velocity"



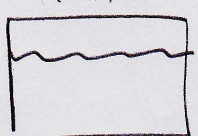
Forces balance
acceleration $= 0$

Ex: A tank initially contains 100g of salt dissolved in 200L of water.
A 5g/L salt solution is pumped in at 4 L/min. The well-mixed solution is pumped out at 4L/min.
Find a formula for mass of salt in tank.

$m =$ mass (grams)
$t =$ time (mins)

$$5\frac{g}{L} \cdot 4\frac{L}{min} = 20\frac{g}{min}$$

TANK



200L

Mass of salt is changing

$$\frac{m}{200}\frac{g}{L} \cdot 4\frac{L}{min} = \frac{4m}{200}\frac{g}{min} = \frac{m}{50}\frac{g}{min}$$

Rate of change of mass = Inflow − Outflow

$$\frac{dm}{dt} = 20 - \frac{m}{50}$$

$\boxed{\text{Liner}}$

1) Standard Form

$$\frac{dm}{dt} + \frac{m}{50} = 20$$

$$\boxed{dm} + \boxed{m}\frac{}{50}\,dt = 20\,dt$$

2) I.F.

$P(t) = \frac{1}{50}$

$\int P(t)\,dt = \frac{t}{50}$

I.F. $= e^{t/50}$

3) $e^{t/50}\,dm + \frac{1}{50}e^{t/50}\,m\,dt = 20e^{t/50}\,dt$

$$d(e^{t/50}\cdot m) = 20e^{t/50}\,dt$$

4) Integrate

$$e^{t/50}\cdot m = 20(50)e^{t/50} + C$$

$$me^{t/50} = 1000e^{t/50} + C$$

Mult. by $e^{-t/50}$: $\boxed{m = 1000 + Ce^{-t/50}}$

$t=0$, $m=100$: $100 = 1000 + C$

$C = -900$

$$\boxed{m = 1000 - 900e^{-t/50}}$$

As $t \to \infty$, what happens to m?

$$-900e^{-t/50} \to 0$$

$$m \to 1000 \text{ grams}$$

$$\boxed{\begin{array}{l}\text{Note: When } m = 1000 \text{ grams} \\ \quad \text{Inflow} = 20\text{g/min} = \text{outflow}\end{array}}$$