31.4 First-Order Linear DE's

1st order linear DE

$$dy + P(x)\,y\,dx = Q(x)\,dx$$

where $P(x), Q(x)$ depend only on x (not on y)

Alternative forms:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$y' + P(x)y = Q(x)$$

Ex: Is it a first-order linear DE?

a) $y' + \frac{y}{x^2} = x^3 + 1$ $\boxed{\text{Yes}}$ $P(x) = \frac{1}{x^2}$ $Q(x) = x^3 + 1$

b) $y' + y = 2$ $\boxed{\text{Yes}}$ $P(x) = 1$ $Q(x) = 2$
don't depend on y ✓

c) $y' + x^3 y^2 = 1$ $\boxed{\text{No}}$ y^2 should be y

d) $xy' + y = \ln x$

$y' + \frac{y}{x} = \frac{\ln x}{x}$ $\boxed{\text{Yes}}$ $P(x) = \frac{1}{x}$ $Q(x) = \frac{\ln x}{x}$

e) $yy' + y = y \ln x$

$y' + 1 = \ln x$ $\boxed{\text{No}}$ 1 should be y

Product Rule for Differentials

$$d(e^{-4x}y) = e^{-4x}dy + y(-4e^{-4x}dx)$$

$$= e^{-4x}dy - 4e^{-4x}y\,dx$$

Ex: Write as a differential

a) $\underbrace{e^{-4x}}dy - 4e^{-4x}y\,dx$

$= d(\underbrace{e^{-4x}}y)$ $\boxed{\text{shortcut}}$

b) $x^4 dy + 4x^3 y\,dx$

$= d(x^4 y)$

c) $e^x dy + e^x y\,dx$

$= d(e^x y)$

d) $\ln x\,dy + \frac{y}{x}dx$

$= d(\ln x\,(y))$

or $d(y \ln x)$

To solve a first-order linear DE :

Use the <u>integrating factor</u> $e^{\int P(x)dx}$

<u>Ex</u>: Solve $\frac{dy}{y} - 4dx = \frac{e^{6x}}{y} dx$

1) Standard Form

$$dy + P(x)\,y\,dx = Q(x)\,dx$$

$$\boxed{dy - 4y\,dx = e^{6x}\,dx}$$

2) Integrating Factor

$P(x) = -4$

$\int P(x)\,dx = -4x \leftarrow \boxed{\text{omit } +C}$

$$\text{I.F.} = e^{\int P(x)dx} = e^{-4x}$$

3) Multiply Standard Form by I.F.

$$e^{-4x}dy - 4e^{-4x}y\,dx = e^{-4x}\cdot e^{6x}\,dx$$

$$\boxed{\text{Left Side will be a differential}}$$

$$d(e^{-4x}y) = e^{2x}\,dx$$

4) Integrate

$$\int d(e^{-4x}y) = \int e^{2x}\,dx$$

$$\boxed{e^{-4x}y = \frac{e^{2x}}{2} + C}$$

explicit solution:

~~Divide by e^{-4x}~~

Mult. by e^{4x} : $\boxed{y = \frac{e^{6x}}{2} + Ce^{4x}}$

Ex: Solve $\cos x \frac{dy}{dx} = 7 - y\sin x$

if $y(\frac{\pi}{3}) = 4$

1) Standard Form

$$\cos x\,dy = 7\,dx - y\sin x\,dx$$

$$dy = \frac{7}{\cos x}dx - \frac{y\sin x}{\cos x}dx$$

$$\boxed{dy + y\tan x\,dx = 7\sec x\,dx}$$

2) I.F.

$P(x) = \tan x$

$\int P(x)\,dx = \ln \sec x$

$e^{\int P(x)\,dx} = e^{\ln \sec x}$

$= \sec x$

3) Multiply standard form by I.F.

$\sec x\,dy + y \sec x \tan x\,dx = 7\sec^2 x\,dx$

$d(\sec x \cdot y) = 7\sec^2 x\,dx$

4) Integrate

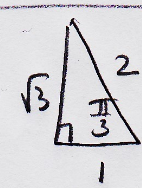
$\int d(\sec x \cdot y) = \int 7\sec^2 x\,dx$

$\boxed{\sec x \cdot y = 7\tan x + C}$

5) Find C

$x = \frac{\pi}{3}$, $y = 4$: $2(4) = 7\sqrt{3} + C$

$C = 8 - 7\sqrt{3}$



$\sec \frac{\pi}{3} = 2$

$\tan \frac{\pi}{3} = \sqrt{3}$

$\sec x \cdot y = 7\tan x + 8 - 7\sqrt{3}$

or $\dfrac{y}{\cos x} = \dfrac{7\sin x}{\cos x} + 8 - 7\sqrt{3}$

$$\boxed{y = 7\sin x + (8 - 7\sqrt{3})\cos x}$$

<u>Ex</u>: Solve $\dfrac{dx}{dt} + 2tx = 3t$ with $x(0) = 2$

1) Standard Form

$$\begin{cases} \boxed{dy} + P(x)\boxed{y}\,dx = Q(x)\,dx \\ dx + P(t)\,x\,dt = Q(t)\,dt \end{cases}$$

$\boxed{\begin{array}{l} y \to x \\ x \to t \end{array}}$

$$\boxed{dx + 2tx\,dt = 3t\,dt}$$

2) I.F.

$P(t) = 2t$

$\int P(t)\,dt = t^2$

$e^{\int P(t)\,dt} = e^{t^2}$

3) Multiply Standard Form by I.F.

$$e^{t^2}dx + 2te^{t^2}x\,dt = 3te^{t^2}dt$$

$$d(e^{t^2}x) = 3te^{t^2}dt$$

4) Integrate

$$\int d(e^{t^2}x) = \frac{3}{2}\int 2te^{t^2}dt$$

$$\boxed{e^{t^2}x = \frac{3}{2}e^{t^2} + C}$$

5) Find C using $x(0)=2$

$x=2$
$t=0$:

$$1(2) = \frac{3}{2}(1) + C$$

$$\frac{4}{2} = \frac{3}{2} + C$$

$$C = \frac{1}{2}$$

$$e^{t^2}x = \frac{3}{2}e^{t^2} + \frac{1}{2}$$

or $$x = \frac{3}{2} + \frac{1}{2}e^{-t^2}$$

<u>Ex</u>: Solve $x\,dy + y^5\,dy = y\,dx$

1) Standard Form

$$\begin{cases} dy + P(x)y\,dx = Q(x)\,dx \\ \quad \text{Not linear in } y \\ dx + P(y)x\,dy = Q(y)\,dy \end{cases}$$

$x \leftrightarrow y$

$$y\,dx = x\,dy + y^5\,dy$$

$$y\,dx - x\,dy = y^5\,dy$$

$$\boxed{dx - \frac{x}{y}\,dy = y^4\,dy}$$

2) I.F.

$$P(y) = -\frac{1}{y}$$

$$\int P(y)\,dy = -\ln y$$

$$e^{\int P(y)\,dy} = e^{-\ln y}$$

$$= e^{\ln y^{-1}}$$

$$= y^{-1}$$

3) Multiply Standard Form by I.F.

$$y^{-1}dx - xy^{-2}dy = y^3\,dy$$

$$d(y^{-1}x) = y^3\,dy$$

4) Integrate

$$y^{-1}x = \frac{y^4}{4} + C$$

Mult. by y: $x = \frac{y^5}{4} + Cy$

RECAP: 31.2 and 31.4

Ex: Identify the method (separable or linear)

a) $dx = 12xy^5\,dy$

$\frac{dx}{x} = 12y^5\,dy$ $\boxed{\text{Separable}}$

b) $dx = 12xy^5\,dy + 9x\,dy$

$dx = (12y^5 + 9)x\,dy$

$\frac{dx}{x} = (12y^5 + 9)\,dy$ $\boxed{\text{Separable}}$

c) $x^2\,dy = 9x^3y\,dx + 8x^7\,dx$

Not separable

$dy = 9xy\,dx + 8x^5\,dx$

$dy - 9xy\,dx = 8x^5\,dx$ $\boxed{\text{Linear}}$

d) $\frac{v}{t} = \ln t - \frac{dv}{dt}$

Not separable

$\frac{dv}{dt} + \frac{v}{t} = \ln t$

$\times\ dt:$ $dv + \frac{v}{t}\,dt = \ln t\,dt$ $\boxed{\text{Linear}}$

$\boxed{dv + P(t)v\,dt = Q(t)\,dt}$ ✓

Ex: Solve $\frac{dy}{dx} = 4x - 2y$

Not separable

Linear ✓

1) Standard Form

$$dy = 4x\,dx - 2y\,dx$$

$$\boxed{dy + 2y\,dx = 4x\,dx}$$

2) I.F.

$$P(x) = 2$$

$$\int P(x)\,dx = 2x$$

$$\text{I.F.} = e^{2x}$$

3) $e^{2x}dy + 2e^{2x}y\,dx = 4xe^{2x}\,dx$

$$d(e^{2x}y) = 4xe^{2x}\,dx$$

4) Integrate

$$\int d(e^{2x}y) = \int 4xe^{2x}\,dx$$

	D	I
+	$4x$	e^{2x}
−	4	$e^{2x}/2$
		$e^{2x}/4$

$$e^{2x}y = 2xe^{2x} - e^{2x} + C$$

or $y = 2x - 1 + Ce^{-2x}$

Ex: Solve $9x^2y\,dy = (x^2-1)^2(y^2+1)\,dx$

$$\frac{9x^2y\,dy}{x^2(y^2+1)} = \frac{(x^2-1)^2(y^2+1)\,dx}{x^2(y^2+1)}$$

$$\frac{9y\,dy}{y^2+1} = \frac{(x^2-1)^2}{x^2}dx \quad \text{Separable}$$

$$\int \frac{9y\,dy}{y^2+1} = \int \frac{(x^2-1)^2}{x^2}dx$$

$$\boxed{\begin{aligned}(x^2-1)^2 &= x^4 - 2x^2 + 1\\ \frac{(x^2-1)^2}{x^2} &= x^2 - 2 + x^{-2}\end{aligned}}$$

$$\frac{9}{2}\int \frac{2y\,dy}{y^2+1} = \int (x^2 - 2 + x^{-2})\,dx$$

$$\frac{9}{2}\ln(y^2+1) = \frac{x^3}{3} - 2x - x^{-1} + C$$