31.2 Separation of Variables

We use $\frac{dy}{dx}$ rather than y' in this section.

Ex: Solve $\quad dx = 12xy^5 dy$

$\div x$: $\quad \frac{dx}{x} = 12y^5 dy$

"Variables are separated"
Not always possible

$$\int \frac{dx}{x} = \int 12y^5 dy$$

$$\ln x + C_1 = 2y^6 + C_2$$

For simplicity, omit absolute values when solving DE

$\cancel{\ln|x| + C_1}$

Shortcut : $\ln x = 2y^6 + \underbrace{C_2 - C_1}_{C}$

Use only <u>one constant</u> when integrating both sides

$$\ln x = 2y^6 + C$$

or $\ln x - 2y^6 = C$

$\boxed{\ln x - 2y^6 = c}$ is an <u>implicit</u> solution

$y = \pm \sqrt[6]{\frac{\ln x - c}{2}}$ " <u>explicit</u> solution

Give implicit solution unless asked for explicit.

<u>Ex</u>: Solve $3\frac{dy}{dx} = \frac{y(x+1)}{x}$

Mult by dx: $3dy = \frac{y(x+1)}{x}dx$

$\div y$: $\frac{3dy}{y} = \frac{x+1}{x}dx$

Variables are separated

$$\int \frac{3}{y}dy = \int \left(1 + \frac{1}{x}\right)dx$$

$$3\ln y = x + \ln x + C$$

- No absolute values
- Only need 1 constant

RECALL $\int \frac{du}{u} = \ln|u| + C$

$$\int \frac{2x}{x^2+5}\,dx = \ln|x^2+5| + C$$

$$\int \frac{3x}{x^2+5}\,dx = \int \frac{\frac{3}{2}(2x)}{x^2+5}\,dx$$

$$= \frac{3}{2}\ln|x^2+5| + C$$

$$n\ln a = \ln a^n$$
$$\ln a + \ln b = \ln(ab)$$
$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$

<u>Ex</u>: Solve $4xy\,dx + (x^2+1)\,dy = 0$

$\div y(x^2+1)$: $$\frac{4xy\,dx}{y(x^2+1)} + \frac{(x^2+1)\,dy}{y(x^2+1)} = \frac{0}{y(x^2+1)}$$

$$\frac{4x\,dx}{x^2+1} + \frac{dy}{y} = 0$$

$$\int \frac{2(2x\,dx)}{x^2+1} + \int \frac{dy}{y} = \int 0$$

$$2\ln(x^2+1) + \ln y = C$$

To get an explicit solution:

$$\ln(x^2+1)^2 + \ln y = c$$

$$\ln[(x^2+1)^2 y] = c$$

$e^{LS} = e^{RS}$:
$$e^{\ln[(x^2+1)^2 y]} = e^{c}$$

$$(x^2+1)^2 y = C_1$$

$$y = \frac{C_1}{(x^2+1)^2}$$

<u>Ex</u>: Solve $y(x^3-5)dy - 6x^2(y^2+4)dx = 0$

$$\frac{y(x^3-5)dy}{(x^3-5)(y^2+4)} - \frac{6x^2(y^2+4)dx}{(x^3-5)(y^2+4)} = 0$$

$$\frac{y\,dy}{y^2+4} - \frac{6x^2\,dx}{x^3-5} = 0$$

$$\int \frac{\frac{1}{2}(2y\,dy)}{y^2+4} - \int \frac{2(3x^2dx)}{x^3-5} = \int 0$$

$$\frac{1}{2}\ln(y^2+4) - 2\ln(x^3-5) = C$$

Alternative answers (by simplifying)

$$\times 2: \ln(y^2+4) - 4\ln(x^3-5) = \cancel{2C}\ C_1$$

$$\ln(y^2+4) + \ln(x^3-5)^{-4} = C_1$$

$$\ln\left[\frac{y^2+4}{(x^3-5)^4}\right] = C_1$$

$e^{LS} = e^{RS}$:

$$\frac{y^2+4}{(x^3-5)^4} = \cancel{e^{C_1}}\ C_2$$

$$y^2+4 = C_2(x^3-5)^4$$

Ex: Solve $\quad e^{2x+t}\,dx + dt = 0$

$$e^{2x}\cdot e^{t}\,dx + dt = 0$$

$\div e^t$: $$e^{2x}\,dx + \frac{dt}{e^t} = 0$$

$$\int e^{2x}\,dx + \int e^{-t}\,dt = \int 0$$

$$\frac{e^{2x}}{2} - e^{-t} = C \quad \checkmark$$

or $$e^{2x} - 2e^{-t} = C_1 \quad \checkmark$$

Ex: Solve $\quad (x^2+1)^3\,dy + 12x\,dx = 0$
explicitly if the solution passes through the point $(1,4)$.

$\div (x^2+1)^3$: $$dy + \frac{12x\,dx}{(x^2+1)^3} = 0$$

$$\int dy + \int \frac{12x\,dx}{(x^2+1)^3} = \int 0 \qquad (A)$$

Substitution

$$u = x^2+1$$
$$du = 2x\,dx$$
$$12x\,dx = 6\,du$$

$$\int \frac{12x\,dx}{(x^2+1)^3} = \int \frac{6\,du}{u^3}$$
$$= \int 6u^{-3}\,du$$
$$= -3u^{-2} + C$$
$$= -3(x^2+1)^{-2} + C$$

(*) $y - 3(x^2+1)^{-2} = C_1$

$$\boxed{y = C_1 + 3(x^2+1)^{-2}}$$ explicit

Sub $x=1$, $y=4$: $4 = C_1 + \frac{3}{2^2}$

$$\frac{16}{4} - \frac{3}{4} = C_1$$

$$C_1 = \frac{13}{4}$$

$$\boxed{y = \frac{13}{4} + 3(x^2+1)^{-2}}$$

Ex: Solve $4y\cos y\,dy - \sin y\,dy = y\sin y\,dx$

given $y = \frac{\pi}{2}$ when $x = 0$

$$(4y\cos y - \sin y)\,dy = y\sin y\,dx$$

$$\frac{4y\cos y - \sin y}{y\sin y}\,dy = dx$$

$$\left(\frac{4y\cos y}{y\sin y} - \frac{\sin y}{y\sin y}\right)dy = dx$$

$$\int\left(4\cot y - \frac{1}{y}\right)dy = \int dx$$

$$\boxed{-4\ln\csc y - \ln y = x + C}$$

Sub $x=0$, $y=\frac{\pi}{2}$: $-4\ln(1) - \ln\left(\frac{\pi}{2}\right) = C$

$$-\ln\left(\frac{\pi}{2}\right) = C$$

$$-4\ln\csc y - \ln y = x - \ln\left(\frac{\pi}{2}\right)$$

Alternatively (by simplifying)

$$-4\ln \csc y - \ln y + \ln\left(\frac{\pi}{2}\right) = x$$

$$\ln(\csc y)^{-4} + \ln\left(\frac{\pi}{2y}\right) = x$$

$$\ln \sin^4 y + \ln\left(\frac{\pi}{2y}\right) = x$$

$$\boxed{\begin{aligned}(\csc y)^{-1} &= \sin y\\ (\csc y)^{-4} &= \sin^4 y\end{aligned}}$$

$$\ln\left[\frac{\pi \sin^4 y}{2y}\right] = x$$

$e^{LS} = e^{RS}$: $\dfrac{\pi \sin^4 y}{2y} = e^x$ ✓

<u>Ex</u>: Solve $y \ln x \, dx = (1 + \ln y)\,dy$

$$\ln x \, dx = \frac{1+\ln y}{y}\,dy$$

$$\int \ln x \, dx = \int \frac{1+\ln y}{y}\,dy \quad ☆$$

Integration by Parts

Substitution

$\int \ln x \, dx$:

	D	I	
(u)	$\ln x$	1	(dv)
(du)	$\frac{1}{x}$	x	(v)

$$\int u \, dv = uv - \int v \, du$$

$$\int \ln x \, dx = x \ln x - \int 1 \, dx$$

$$= x \ln x - x + C$$

$$\int \frac{1 + \ln y}{y} \, dy$$

$$u = 1 + \ln y$$

$$du = \frac{1}{y} \, dy$$

$$\text{Integral} = \int u \, du$$

$$= \frac{u^2}{2} + C$$

$$= \frac{1}{2}(1 + \ln y)^2 + C$$

☆ $$x \ln x - x = \frac{1}{2}(1 + \ln y)^2 + C_1$$