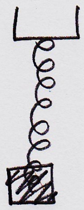
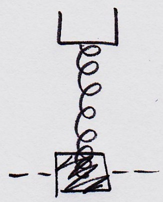
31.10 Applications of Higher-Order DE

Spring-Mass Systems



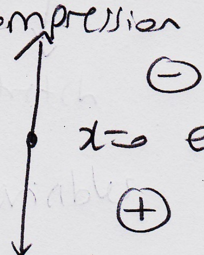
Vertical Motion



equilibrium position: where F_g and restorative force balance

Variables: x = displacement (m)
t = time (s)

compression



$x=0$ equilibrium position

stretch

Mass · Acceleration = Net Force

$$\boxed{m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + f(t) \qquad (b, k > 0)}$$

☆

damping ∝ velocity,
in opposite direction
e.g. friction / air resistance

Hooke's Law force
∝ displacement,
in opposite direction

external force

<u>Ex</u>: $m = 1\text{kg}$ $b = 2\ N/(m/s)$ $k = 4\ N/m$ $f(t) = 0$

Find the equation of motion.

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + f(t)$$

$$\frac{d^2x}{dt^2} = -2\frac{dx}{dt} - 4x$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 4x = 0$$

Auxiliary equation: use n

$$n^2 + 2n + 4 = 0$$

$$n = \frac{-2 \pm \sqrt{4-16}}{2}$$

$$n = \frac{-2 \pm \sqrt{-12}}{2} \leftarrow \sqrt{4}\sqrt{3}\sqrt{-1}$$

$$n = \frac{-2 \pm 2\sqrt{3}\, j}{2}$$

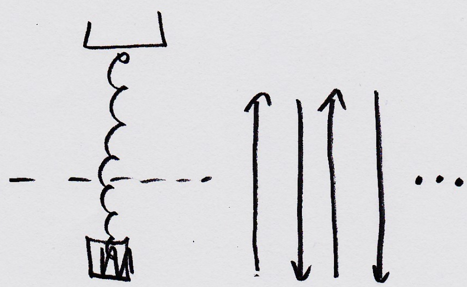
$$n = -1 \pm \sqrt{3}\, j \qquad (\alpha = -1, \beta = \sqrt{3})$$

$$\cancel{y = e^{-x}(\qquad)}$$

x is a function of t

$$x = e^{-t}(C_1 \sin\sqrt{3}\,t + C_2 \cos\sqrt{3}\,t)$$

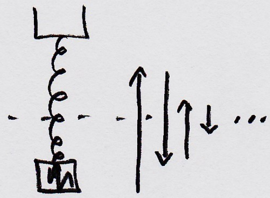
If $b = 0$ and $f(t) = 0$ we have harmonic motion



If $f(t)=0$ and $b \neq 0$ there are three scenarios:

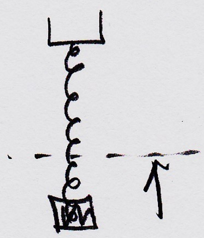
I) $\boxed{\text{2 complex roots}}$

"underdamped motion": oscillation but amplitude decreases



II) $\boxed{\text{distinct real roots}}$

"overdamped motion": no oscillation



III) $\boxed{\text{repeated real roots}}$

"critically damped motion"

No oscillation, but any decrease in b causes underdamped motion (oscillation)

Ex: A 49N weight stretches a certain spring 2.0 cm. The spring is pulled 20cm longer than its equilibrium length and released. There is no damping or external force. Find equation of motion.

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + f(t)$$

$$\boxed{m = \frac{49N}{9.8\ m/s^2} = 5\ kg}$$

$$\boxed{b=0}$$

$$\boxed{f(t)=0}$$

$$\boxed{\begin{aligned} 49N &= (0.02m)k \\ k &= 2450\ N/m \end{aligned}}$$

Initial Conditions: $x = 0.2m$ when $t=0$
$x' = 0$ when $t=0$

$$5\frac{d^2x}{dt^2} = -2450x$$

$$5\frac{d^2x}{dt^2} + 2450x = 0$$

$$5n^2 + 2450 = 0$$

$$n^2 = -\frac{2450}{5}$$

$$n^2 = -490$$

$n = \pm\sqrt{-490} \leftarrow \sqrt{490}\sqrt{-1}$

$n \simeq \pm 22j \qquad (\alpha = 0,\ \beta = 22)$

$\boxed{\text{OK to round in Section 31.10}}$

$$\boxed{x = C_1 \sin 22t + C_2 \cos 22t}$$

$\boxed{\begin{matrix} x = 0.2 \\ t = 0 \end{matrix}}$: $\quad 0.2 = C_2$

$$\boxed{x = C_1 \sin 22t + 0.2 \cos 22t}$$

$x' = 22C_1 \cos 22t - 4.4 \sin 22t$

$\boxed{\begin{matrix} x' = 0 \\ t = 0 \end{matrix}}$: $\quad 0 = 22C_1$

$C_1 = 0$

$x = 0.2 \cos 22t$

<u>Ex</u>: $m = 5.0\ kg$ $\quad k = 2450\ N/m$

$f(t) = 0$

How much damping causes critically damped motion?

$\boxed{\text{Find } b \text{ so that auxiliary equation has repeated real roots.}}$

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + f(t)$$

$$5\frac{d^2x}{dt^2} + b\frac{dx}{dt} + 2450x = 0$$

$$5r^2 + br + 2450 = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 49000}}{10}$$

Roots will be repeated when

$$\sqrt{b^2 - 49000} = 0$$

$$b^2 - 49000 = 0$$

$$b^2 = 49000$$

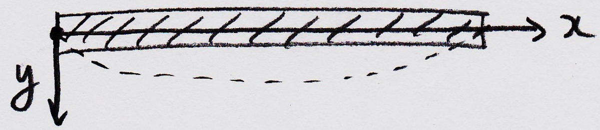
$$b = \pm\sqrt{49000}$$

$$b = \pm 220$$

But $b > 0$

$\boxed{b = 220} \rightarrow$ Damping approx. 220 times velocity, in opposite direction.

Deflection of a Beam



x = distance from left end
y = deflection

$$EI \frac{d^4y}{dx^4} = \underbrace{w(x)}$$

Constant stiffness of beam material

load distribution along beam

Ex: Beam of length L has constant w (due to its own weight). Find deflection if

at $x=0$: $y=0$ and $y''=0$
$x=L$: same

$$EI \frac{d^4y}{dx^4} = w$$

$$\frac{d^4y}{dx^4} = \frac{w}{EI} \leftarrow \text{Call this } k$$

$$\frac{d^4y}{dx^4} = k$$

Nonhomogeneous DE

1) y_c

$$\frac{d^4y}{dx^4} = 0$$

$$m^4 = 0$$

$$m = 0,0,0,0$$

$$y_c = (C_1 + C_2 x + C_3 x^2 + C_4 x^3)e^{0x}$$

$$\boxed{y_c = C_1 + C_2 x + C_3 x^2 + C_4 x^3}$$

2) y_p

$f(x) = k$

$y_p = A$ won't work   BAD CASE

$$\boxed{y_p = Ax^4}$$

3) $y_p \to$ DE

$$\left.\begin{aligned} y_p &= Ax^4 \\ y_p' &= 4Ax^3 \\ y_p'' &= 12Ax^2 \\ y_p''' &= 24Ax \\ y_p^{(4)} &= 24A \end{aligned}\right\} \to \text{DE}$$

DE : $\frac{d^4y}{dx^4} = k$

$24A = k$

$A = \frac{k}{24}$

$$\boxed{y_p = \frac{k}{24}x^4}$$

4) $y = y_c + y_p$

$y = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + \frac{k}{24}x^4$

5) Find C_1, C_2, C_3, C_4

$\boxed{\begin{array}{l} x=0 \\ y=0 \end{array}}$: $0 = C_1 \rightarrow$

$$\boxed{y = C_2 x + C_3 x^2 + C_4 x^3 + \frac{k}{24}x^4}$$

$y' = C_2 + 2C_3 x + 3C_4 x^2 + \frac{4k}{24}x^3$

$y'' = 2C_3 + 6C_4 x + \frac{12k}{24}x^2$

$\boxed{\begin{array}{l} x=0 \\ y''=0 \end{array}}$: $0 = 2C_3$

$C_3 = 0 \rightarrow$

$$\boxed{y = C_2 x + C_4 x^3 + \frac{k}{24}x^4}$$

$\boxed{\begin{array}{l} x=L \\ y=0 \end{array}}$: $$0 = C_2 L + C_4 L^3 + \frac{k}{24} L^4 \quad (1)$$

$$y' = C_2 + 3C_4 x^2 + \frac{4k}{24} x^3$$

$$y'' = 6C_4 x + \frac{12k}{24} x^2$$

$\boxed{\begin{array}{l} x=L \\ y''=0 \end{array}}$: $$0 = 6C_4 L + \frac{12k}{24} L^2 \quad (2)$$

(2) : $$6C_4 L = -\frac{12k}{24} L^2$$

$$\boxed{C_4 = -\frac{2k}{24} L} \rightarrow (1)$$

(1) : $$0 = C_2 L - \frac{2k}{24} L^4 + \frac{k}{24} L^4$$

$$C_2 L = \frac{2k}{24} L^4 - \frac{k}{24} L^4$$

$$C_2 L = \frac{k}{24} L^4$$

$$\boxed{C_2 = \frac{k}{24} L^3}$$

$C_2, C_4 \rightarrow y$ $$y = \frac{k}{24} L^3 x - \frac{2k}{24} L x^3 + \frac{k}{24} x^4$$

$$y = \frac{k}{24} \left(L^3 x - 2L x^3 + x^4 \right)$$

$$\boxed{y = \frac{w}{24EI} \left(L^3 x - 2L x^3 + x^4 \right)}$$