INTEGRAL REVIEW

Evaluate:

a) $\int \sin x \, dx$

$= -\cos x + C$

b) $\int \cos x \, dx$

$= \sin x + C$

c) $\int \sin(4x) \, dx$

$= -\frac{1}{4}\cos(4x) + C$

$$\boxed{\begin{array}{l}\text{Shortcut:} \\ \int \sin(kx)\,dx \\ \quad = -\frac{1}{k}\cos(kx) + C\end{array}}$$

d) $\int e^{2x} \, dx$

$= \frac{e^{2x}}{2} + C$

e) $\int \frac{1}{x}\, dx$

$= \ln|x| + C$

f) $\int \frac{6x}{x^2+1}\, dx$

$= 3\int \frac{du}{u}$

$= 3\ln|u| + C$

$= 3\ln|x^2+1| + C$

$$\boxed{\begin{aligned} u &= x^2+1 \\ du &= 2x\,dx \\ 6x\,dx &= 3\,du \end{aligned}}$$

g) $\int \frac{1}{x^6}\, dx$

$= \int x^{-6}\, dx$

$= -\frac{1}{5} x^{-5} + C$

h) $\int \frac{x^2}{(x^3+1)^4}\, dx$

$= \frac{1}{3}\int \frac{du}{u^4}$

$= \frac{1}{3}\int u^{-4}\, du$

$= -\frac{1}{9} u^{-3} + C$

$= -\frac{1}{9}(x^3+1)^{-3} + C$

$$\boxed{\begin{aligned} u &= x^3+1 \\ du &= 3x^2\,dx \\ x^2\,dx &= \frac{du}{3} \end{aligned}}$$

DERIVATIVE REVIEW

Find $f'(x)$:

a) $f(x) = \frac{1}{x^2}$ $= x^{-2}$

$f'(x) = -2x^{-3}$

b) $f(x) = (x^3 + 1)^4$

$f'(x) = 4(x^3+1)^3 \, 3x^2$

$= 12x^2(x^3+1)^3$

c) $f(x) = e^{6x}$

$f'(x) = 6e^{6x}$

d) $f(x) = \ln(2x + 1)$

$f'(x) = \frac{1}{2x+1} \cdot 2 = \frac{2}{2x+1}$

e) $f(x) = \sin(4x)$

$f'(x) = 4\cos(4x)$

f) $f(x) = \cos x^2$

$f'(x) = -2x\sin x^2$

g) $f(x) = \tan(9x+1)$

$f'(x) = 9\sec^2(9x+1)$

h) $f(x) = \sec(3x)$

$f'(x) = 3\sec 3x \tan 3x$

i) $f(x) = \csc(2x)$

$f'(x) = -2\csc 2x \cot 2x$

j) $f(x) = \cot x^3$

$f'(x) = -3x^2 \csc^2 x^3$

k) $f(x) = \sin^{-1} x$

$f'(x) = \dfrac{1}{\sqrt{1-x^2}}$

$\boxed{\text{Formula sheet } a=1 \quad \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C}$

l) $f(x) = \tan^{-1} x$

$f'(x) = \dfrac{1}{1+x^2}$

$\boxed{\text{Formula Sheet } a=1 \quad \int \frac{dx}{1+x^2} = \tan^{-1}x + C}$

m) $f(x) = uv$

$f'(x) = uv' + vu'$

n) $f(x) = \frac{u}{v}$

$f'(x) = \dfrac{vu' - uv'}{v^2}$

MIXED REVIEW

Evaluate:

a) $\int e^{9x}\,dx$

$= \frac{e^{9x}}{9} + C$

b) $\frac{d}{dx}[e^{9x}]$

$= 9e^{9x}$

c) $\int \cos(2x)\,dx$

$= \frac{\sin 2x}{2} + C$

d) $\frac{d}{dx}[\cos(2x)]$

$= -2\sin 2x$

e) $\int \frac{5}{x}\,dx$

$= 5\ln|x| + C$

f) $\frac{d}{dx}[\frac{5}{x}]$ $= \frac{d}{dx}[5x^{-1}]$

$= -5x^{-2}$

g) $\int xe^{2x}\,dx$

$= \boxed{\frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C}$

Integration by Parts

	D	I
$+$	x	e^{2x}
$-$	1	$\frac{e^{2x}}{2}$
		$\frac{e^{2x}}{4}$

31.1 Solutions of Differential Equations

<u>Differential equation (DE)</u>: an equation that contains at least one derivative

e.g. $y'' + 16y = 0$

$6y''' = y'$

We'll focus on:

- solving DE's
- modelling real-life situations
  e.g. spring-mass systems
  deflection of beams

The <u>order</u> of a DE is the order of the highest derivative in the DE

<u>Ex</u>:

$y'' + 16y = 0$ — 2nd order

$6y''' = y'$ — 3rd order

$2x^3y' - 4y = 0$ — 1st order

<u>Notation</u>: Recall y' can be written $\frac{dy}{dx}$

y'' $\qquad \frac{d^2y}{dx^2}$

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0$$

means exactly the same thing as

$$x^2y'' - 2xy' - 4y = 0$$

<u>Ex</u>: a) Check that $\boxed{y = 7e^{-x/2}}$ ← solution

is a solution to $\boxed{2y' + y = 0}$ ← DE

$$\begin{cases} y = 7e^{-x/2} \\ y' = -\frac{7}{2}e^{-x/2} \end{cases}$$

$$\begin{aligned} \text{LS of DE} &= 2y' + y \\ &= 2\left(-\tfrac{7}{2}e^{-x/2}\right) + 7e^{-x/2} \\ &= -7e^{-x/2} + 7e^{-x/2} \\ &= 0e^{-x/2} \\ &= 0 \\ &= \text{RS of DE} \end{aligned}$$

b) Show that $y = x^2$ is not a solution

$$\begin{cases} y = x^2 \\ y' = 2x \end{cases}$$

$$2y' + y = 4x + x^2 \neq 0$$

Most functions are not solutions

Ex: Check that $y = C\sin 4x$ (C=constant)

solves $y'' + 16y = 0$

$$\begin{cases} y = C\sin 4x \\ y' = 4C\cos 4x \\ y'' = -16C\sin 4x \end{cases}$$

$$\begin{aligned} LS &= y'' + 16y \\ &= -16C\sin 4x + 16C\sin 4x \\ &= 0 \\ &= RS \end{aligned}$$

Ex: Check that $y = 5\tan 5x$
is a solution of $y' = 25 + y^2$

$$\begin{cases} y = 5\tan 5x \\ y' = 25\sec^2 5x \end{cases}$$

$\boxed{\text{Start with more complicated side}}$

$$\begin{aligned} RS &= 25 + y^2 \\ &= 25 + (5\tan 5x)^2 \\ &= 25 + 25\tan^2 5x \\ &= 25(1 + \tan^2 5x) \qquad \boxed{1 + \tan^2\theta = \sec^2\theta} \\ &= 25\sec^2 5x \\ &= y' \\ &= LS \end{aligned}$$

A solution that has
\# unknown constants = order
is called the <u>general solution</u>. Otherwise
it's a <u>particular solution</u>.

<u>Ex</u>: $y = C_1 \sin 4x + C_2 \cos 4x$
is the general solution to $y'' + 16y = 0$.

constants: C_1, C_2 2^{nd} order DE
constants = order ✓

Some particular solutions:

$y = -3 \sin 4x$ (0 constants)

$y = 8 \cos 4x$ "

$y = 2 \sin 4x - 7 \cos 4x$ "

$y = C \sin 4x - 9 \cos 4x$ (1 constant)

<u>Ex</u>: a) Check that $y = C_1 x^{-1} + C_2 x^4$
is a solution to $x^2 y'' - 2x y' - 4y = 0$

$$\begin{cases} y = C_1 x^{-1} + C_2 x^4 \\ y' = -C_1 x^{-2} + 4 C_2 x^3 \\ y'' = 2 C_1 x^{-3} + 12 C_2 x^2 \end{cases}$$

$LS = x^2y'' - 2xy' - 4y$

$= x^2(2C_1x^{-3} + 12C_2x^2)$

$-2x(-C_1x^{-2} + 4C_2x^3)$

$-4(C_1x^{-1} + C_2x^4)$

$= 2C_1x^{-1} + 12C_2x^4$

$+2C_1x^{-1} - 8C_2x^4$

$-4C_1x^{-1} - 4C_2x^4$

$= 0C_1x^{-1} + 0C_2x^4$

$= 0$

$= RS$ ✓

b) Is it the general solution?

$\boxed{\text{Yes}}$ # constants = 2 = order

c) List 3 particular solutions

$y = 0$

$y = 8x^{-1} - 5x^4$

$y = Cx^4$

More about Constants

$C_1 + C_2 x + C_3 x$ should be rewritten

$= C_1 + (C_2 + C_3)x$

$= C_1 + C_4 x \Leftarrow$ 2 constants

$\boxed{\text{Always collect like terms!}}$

Ex: Simplify

$A + 1 + C_1 - 5x^2 + C_2 x^2 + C_3 \ln x$

$= (A + 1 + C_1) + (C_2 - 5)x^2 + C_3 \ln x$

$= C_4 + C_5 x^2 + C_3 \ln x \Leftarrow$ 3 constants

Ex: The general solution of $y' - 3y = 6$
is $y = -2 + Ce^{3x}$
Find the particular solution if $\boxed{y(0) = 7}$

means solution passes through $(x, y) = (0, 7)$

$\boxed{y = -2 + Ce^{3x}}$ general solution

Sub $\begin{matrix} x = 0 \\ y = 7 \end{matrix}$: $7 = -2 + Ce^{0}$

$7 = -2 + C$

$9 = C$

$\boxed{y = -2 + 9e^{3x}}$ particular solution

Ex: Show that $\boxed{xy - Cx - Cy = 0}$ solution

solves $\boxed{x^2 y' = -y^2}$ DE

Get y, y' using the solution

$$xy - Cy = Cx$$

$$(x-C)y = Cx$$

$$\boxed{y = \frac{Cx}{x-C}}$$

$$y' = \frac{(x-C)C - Cx(1)}{(x-C)^2}$$

$$= \frac{-C^2}{(x-C)^2}$$

$$\boxed{\begin{aligned} LS &= x^2 y' \\ &= \frac{-C^2 x^2}{(x-C)^2} \end{aligned}}$$

$$\boxed{\begin{aligned} RS &= -y^2 \\ &= -\left(\frac{Cx}{x-C}\right)^2 \\ &= -\frac{C^2 x^2}{(x-C)^2} \end{aligned}}$$

LS = RS ✓