29.4

Double Integrals

Ex: $$\int_1^2 \int_0^1 (x^2 + xy^3)\, dx\, dy$$

Inside integral first
dx: Integrate wrt x (y is constant)

$$= \int_1^2 \left[\frac{x^3}{3} + \frac{x^2}{2} y^3 \right]_{x=0}^{x=1} dy$$

$$= \int_1^2 \left[\left(\frac{1}{3} + \frac{y^3}{2} \right) - 0 \right] dy$$

$$= \int_1^2 \left(\frac{1}{3} + \frac{y^3}{2} \right) dy$$

$$= \left[\frac{y}{3} + \frac{y^4}{8} \right]_1^2$$

$$= \left(\frac{2}{3} + \frac{16}{8} \right) - \left(\frac{1}{3} + \frac{1}{8} \right)$$

$$= \frac{16}{24} + \frac{48}{24} - \left(\frac{8}{24} + \frac{3}{24} \right)$$

$$= \frac{53}{24}$$

<u>Ex:</u>

$$\underbrace{\int_0^2 \int_0^{x/2} x y^2 \, dy}_{\text{Integrate wrt } y \ (x \text{ is constant})} \, dx$$

$$= \int_0^2 \left[\frac{x y^3}{3} \right]_{y=0}^{y=x/2} dx$$

$$= \int_0^2 \left[\frac{x}{3}\left(\frac{x}{2}\right)^3 - 0 \right] dx$$

$$= \int_0^2 \frac{x}{3}\left(\frac{x^3}{8}\right) dx$$

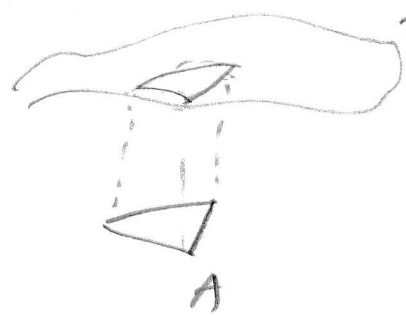
$$= \int_0^2 \frac{x^4}{24} dx$$

$$= \left[\frac{x^5}{120} \right]_0^2$$

$$= \frac{2^5}{120} - 0$$

$$= \frac{32}{120} \text{ or } \frac{4}{15}$$

Volume under a surface z over a region A:



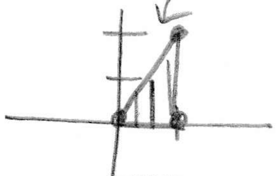
$$V = \iint z \, dx \, dy$$

or $$V = \iint z \, dy \, dx$$

<u>Ex</u>: Find volume under $z = xy$ over the region bounded by $(x,y) = (0,0), (1,0), (1,2)$.

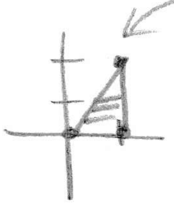
1) Limits on x and y

$y = 2x \quad (y = mx + b)$



$$\boxed{\begin{array}{ll} 0 \le x \le 1 & \text{independent} \\ 0 \le y \le 2x & \text{dependent} \end{array}}$$

OR



$y = 2x$

$\frac{y}{2} = x$

$x = \frac{y}{2}$

$$\boxed{\begin{array}{ll} 0 \le y \le 2 & \text{Ind.} \\ \frac{y}{2} \le x \le 1 & \text{Dep.} \end{array}}$$

2) Either way, <u>dependent variable goes inside</u>

$$V = \int_0^1 \int_0^{2x} xy \, dy \, dx$$

or $$V = \int_0^2 \int_{\frac{y}{2}}^1 xy \, dx \, dy$$

Let's use $$V = \int_0^1 \int_0^{2x} xy \, dy \, dx$$

$$= \int_0^1 \left[\frac{xy^2}{2} \right]_{y=0}^{y=2x} dx$$

$$= \int_0^1 \left[\frac{x(2x)^2}{2} - 0 \right] dx$$

$$= \int_0^1 \left[\frac{x(4x^2)}{2} - 0 \right] dx$$

$$= \int_0^1 2x^3 \, dx$$

$$= \left[\frac{x^4}{2} \right]_0^1$$

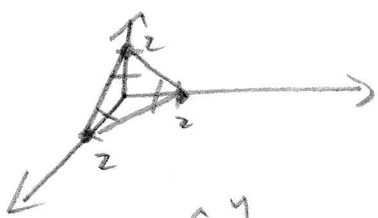
$$= \frac{1}{2} - 0$$

$$= \frac{1}{2}$$

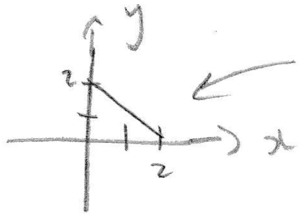
Ex: Volume under plane $x+y+z=2$
in the first octant $(x \geq 0,\ y \geq 0,\ z \geq 0)$?

1) Limits on x and y

plane:



Area:



$y = 2 - x \quad (y = mx + b)$

$$\boxed{\begin{array}{ll} 0 \leq x \leq 2 & \text{Ind.} \\ 0 \leq y \leq 2 - x & \text{Dep.} \end{array}}$$

2) $V = \iint z \, dy \, dx$ Dep. variable inside

$x + y + z = 2$

$\boxed{z = 2 - x - y}$

$$V = \underbrace{\int_0^2 \underbrace{\int_0^{2-x} (2 - x - y) \, dy}_{} \, dx}_{}$$

$$= \int_0^2 \left(2y - xy - \frac{y^2}{2}\right)\Big|_{y=0}^{y=2-x} dx$$

$$= \int_0^2 \left[\left[2(2-x) - x(2-x) - \frac{(2-x)^2}{2}\right] - 0\right] dx$$

$$= \int_0^2 \left[4 - 2x - 2x + x^2 - \frac{(4-4x+x^2)}{2}\right] dx$$

$$= \int_0^2 \left[4 - 4x + x^2 - 2 + 2x - \frac{x^2}{2}\right] dx$$

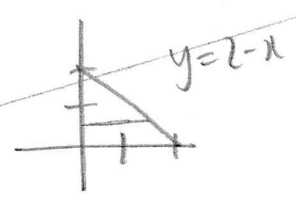
$$= \int_0^2 \left(2 - 2x + \frac{x^2}{2}\right) dx$$

$$= \left[2x - x^2 + \frac{x^3}{6}\right]_0^2$$

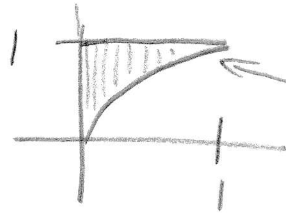
$$= 4 - 4 + \frac{8}{6} - 0$$

$$= \frac{8}{6} \text{ or } \frac{4}{3}$$

Alternatively: $V = \int_0^2 \int_0^{2-y} z \, dx \, dy$



Ex: Volume under $z = e^{y^3}$ over the region:



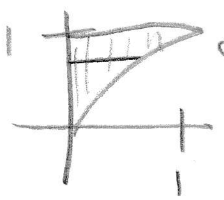
1) Limits $\quad 0 \le x \le 1 \quad$ ind.

$\sqrt{x} \le y \le 1 \quad$ dep.

2) $V = \int\int e^{y^3}\, dy\, dx$ Dep. variable inside

$= \int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\, dy\, dx$

☹ Can't integrate



$y = \sqrt{x}$
$y^2 = x$
$x = y^2$

1) $\boxed{\begin{array}{ll} 0 \le y \le 1 & \text{ind.} \\ 0 \le x \le y^2 & \text{dep.} \end{array}}$

2) $V = \int_0^1 \int_0^{y^2} e^{y^3}\, dx\, dy$

$$= \int_0^1 \left[x e^{y^3} \right]_{x=0}^{x=y^2} dy$$

$$= \int_0^1 \left(y^2 e^{y^3} - 0 \right) dy$$

$$= \int_0^1 y^2 e^{y^3} dy$$

$$= \left[\frac{e^{y^3}}{3} \right]_0^1$$

$$= \frac{e^1}{3} - \frac{e^0}{3}$$

$$= \frac{e-1}{3}$$

Sub $u = y^3$

$du = 3y^2\, dy$

$\frac{du}{3} = y^2\, dy$

$$\int_{y=0}^{y=1} \frac{e^u\, du}{3}$$

$$= \left[\frac{e^u}{3} \right]_{y=0}^{y=1}$$

$$= \left[\frac{e^{y^3}}{3} \right]_0^1$$