28.9 Partial Fractions

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$$\int \frac{1}{2x+3}\,dx$$

$$\boxed{\begin{aligned} u &= 2x+3 \\ du &= 2\,dx \\ dx &= du/2 \end{aligned}}$$

$$= \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|2x+3| + C$$

Shortcut

$$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C$$

Ex: $$\int \frac{5x+1}{x^2+5x}\,dx = \int \frac{5x+1}{x(x+5)}\,dx$$

$$\frac{5x+1}{x(x+5)} = \frac{A}{x} + \frac{B}{x+5} \qquad A, B: \text{constants}$$

"partial fraction expansion"

Multiply by $x(x+5)$:

$$\boxed{5x+1 = A(x+5) + Bx}$$

Sub appropriate values

$x=0$: $1 = 5A$

$A = \frac{1}{5}$

$x=-5$: $-24 = -5B$

$B = \frac{24}{5}$

Conclude $\frac{5x+1}{x(x+5)} = \frac{1}{5} \cdot \frac{1}{x} + \frac{24}{5} \cdot \frac{1}{x+5}$

Integral $= \int \left(\frac{1}{5} \cdot \frac{1}{x} + \frac{24}{5} \cdot \frac{1}{x+5} \right) dx$

$= \frac{1}{5} \ln|x| + \frac{24}{5} \ln|x+5| + C$

$= \frac{1}{5} \left[\ln|x| + 24 \ln|x+5| \right] + C$

Recall $n \ln a = \ln a^n$

$\ln a + \ln b = \ln(ab)$

$= \frac{1}{5} \ln \left| x(x+5)^{24} \right| + C$

Ex: $\int \frac{x+1}{x(x-1)(2x+1)} dx$

$\frac{x+1}{x(x-1)(2x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{2x+1}$

$x+1 = A(x-1)(2x+1) + Bx(2x+1) + Cx(x-1)$

$x=0$: $1 = -A$

$A = -1$

$x=1$: $2 = B(1)(3)$

$B = \frac{2}{3}$

$\boxed{2x+1=0}$

$x = -\frac{1}{2}$: $\frac{1}{2} = C\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)$

$\frac{1}{2} = C\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)$

$$\frac{1}{2} = \frac{3}{4}C$$

$$C = \frac{1}{2}\cdot\frac{4}{3} = \frac{2}{3}$$

$$\text{Integral} = \int\left(\frac{-1}{x} + \frac{2}{3}\cdot\frac{1}{x-1} + \frac{2}{3}\cdot\frac{1}{2x+1}\right)dx$$

$$= -\ln|x| + \frac{2}{3}\ln|x-1| + \cancel{\frac{2}{3}\cdot\frac{1}{2}}\ln|2x+1| + C$$

$$= \frac{1}{3}\left[-3\ln|x| + 2\ln|x-1| + \ln|2x+1|\right] + C$$

$$= \frac{1}{3}\left[\ln|x|^{-3} + \ln|x-1|^2 + \ln|2x+1|\right] + C$$

$$= \frac{1}{3}\ln\left|\frac{(x-1)^2(2x+1)}{x^3}\right| + C$$

Review: Long Division

$$\frac{4x^3-6x^2+11}{2x^2-x+3} = ?$$

$$
\begin{array}{r|l}
 & \quad 2x - 2 \\
\hline
\boxed{2x^2} - x + 3 & \boxed{4x^3} - 6x^2 + 0x + 11 \\
 & -(4x^3 - 2x^2 + 6x) \\
\hline
 & \qquad \boxed{-4x^2} - 6x + 11 \\
 & \quad -(-4x^2 + 2x - 6) \\
\hline
 & \qquad\qquad -8x + 17
\end{array}
$$

$$\frac{4x^3-6x^2+11}{2x^2-x+3} = 2x-2 + \frac{-8x+17}{2x^2-x+3}$$

Ex: $\int \frac{4x^3+5x^2-8x+1}{\dots}$

Ex: $\displaystyle\int \frac{4x^3+5x^2-8x+1}{x^2+x-2}\,dx$

If degree(numerator) $\geq$ degree(denominator) then do long division first

$$\begin{array}{r}
4x+1 \\
x^2+x-2 \enclose{longdiv}{4x^3+5x^2-8x+1} \\
-(4x^3+4x^2-8x) \\
\hline
x^2 \qquad +1 \\
-(x^2+x-2) \\
\hline
-x+3
\end{array}$$

$$\text{Integral} = \int \left(4x+1+\frac{-x+3}{x^2+x-2}\right)dx$$

$$\frac{-x+3}{(x+2)(x-1)} = \frac{A}{x+2}+\frac{B}{x-1}$$

$$\boxed{-x+3 = A(x-1)+B(x+2)}$$

Sub $x=1$: $2=3B$

$B=2/3$

$x=-2$: $5=A(-3)$

$A=-5/3$

$$\text{Integral} = \int\left(4x+1-\frac{5}{3}\cdot\frac{1}{x+2}+\frac{2}{3}\cdot\frac{1}{x-1}\right)dx$$

$$= 2x^2+x-\frac{5}{3}\ln|x+2|+\frac{2}{3}\ln|x-1|+C$$

$$= 2x^2+x+\frac{1}{3}\left[-5\ln|x+2|+2\ln|x-1|\right]+C$$

$$= 2x^2 + x + \frac{1}{3}\left[\ln|x+2|^{-5} + \ln|x-1|^2\right] + C$$

$$= 2x^2 + x + \frac{1}{3}\ln\left|\frac{(x-1)^2}{(x+2)^5}\right| + C$$

Try the "Mixed Integration Problems" on the website