28.7 Integration by Parts

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Ex: $\int x\cos x\,dx$ Previous methods don't work

Let $u = x$ $\quad dv = \cos x\,dx$

$du = dx$ $\quad v = \int \cos x\,dx$

$v = \sin x \cancel{+C}$

$$\int u\,dv = uv - \int v\,du$$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + C$$

Choose u such that du is simpler than u.

Shortcut:

	D	I
$u \to +$	x	$\cos x \leftarrow dv$
$-$	1	$\sin x$
	0	$-\cos x$

$$\int x\cos x\,dx = x\sin x + \cos x + C$$

Ex: $\int_1^2 r^3 e^{2r}\,dr$

	D	I
$+$	r^3	e^{2r}
$-$	$3r^2$	$e^{2r}/2$
$+$	$6r$	$e^{2r}/4$
$-$	6	$e^{2r}/8$
	0	$e^{2r}/16$

$$\int_1^2 r^3 e^{2r}\,dr = \left[\frac{r^3 e^{2r}}{2} - \frac{3r^2 e^{2r}}{4} + \frac{6r e^{2r}}{8} - \frac{6e^{2r}}{16}\right]_1^2$$

$$= \left[4e^4 - 3e^4 + \frac{12e^4}{8} - \frac{6e^4}{16}\right]$$

$$- \left[\frac{1}{2}e^2 - \frac{3e^2}{4} + \frac{6e^2}{8} - \frac{6e^2}{16}\right]$$

$$= \frac{17}{8}e^4 - \frac{1}{8}e^2$$

<u>Ex</u>: $\int 3x\sqrt{1+x}\,dx$

	D	I
+	$3x$	$\sqrt{1+x}$
−	3	$\frac{2}{3}(1+x)^{3/2}$
		$\frac{4}{15}(1+x)^{5/2}$

$\frac{2}{5}\cdot\frac{2}{3}$

$$\int 3x\sqrt{1+x}\,dx = 2x(1+x)^{3/2} - \frac{4}{5}(1+x)^{5/2} + C$$

<u>Ex</u>: $\int 14x^3 \sin x \cos x\,dx$

$2\sin x\cos x = \sin 2x$

$= \int 7x^3 \sin 2x\,dx$

	D	I
+	$7x^3$	$\sin 2x$
−	$21x^2$	$-\cos 2x/2$
+	$42x$	$-\sin 2x/4$
−	42	$\cos 2x/8$
		$\sin 2x/16$

$$= -\frac{7x^3}{2}\cos 2x + \frac{21x^2}{4}\sin 2x + \frac{42x}{8}\cos 2x - \frac{42}{16}\sin 2x + C$$

<u>Ex</u>: $\int x \ln x \, dx$

D	I
~~x~~	~~$\ln x$~~
	~~?~~

D	I
$\ln x$	x
$\frac{1}{x}$	$x^2/2$
$-\frac{1}{x^2}$	$x^3/6$
$\vdots$	$\vdots$

If both columns continue, use $\int u\,dv = uv - \int v\,du$

	D	I	
u	$\ln x$	x	dv
du	$\frac{1}{x}$	$x^2/2$	v

$$\int x \ln x \, dx = \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

Ex: $\int \ln x \, dx$

$$\cancel{\begin{array}{c|c} D & I \\ \hline 1 & \ln x \\ & ? \end{array}}$$

$$\begin{array}{cc|cc} & D & I & \\ \hline u & \ln x & 1 & dv \\ du & \frac{1}{x} & x & v \\ & \vdots & \vdots & \end{array}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \ln x \, dx = x \ln x - \int 1 \, dx$$

$$= x \ln x - x + C$$

Ex: $\int \tan^{-1} x \, dx$

$$\cancel{\begin{array}{c|c} D & I \\ \hline 1 & \tan^{-1} x \\ & ? \end{array}}$$

$$\begin{array}{cc|cc} & D & I & \\ \hline u & \tan^{-1} x & 1 & dv \\ du & \frac{1}{1+x^2} & x & v \\ & \vdots & \vdots & \end{array}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{x}{1+x^2} \, dx$$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx$$

$$= x \tan^{-1} x - \frac{1}{2} \ln|1+x^2| + C$$

Ex: $\int e^x \cos x \, dx$

	D	I	
u	$\cos x$	e^x	dv
	$-\sin x$	e^x	v
du	$\vdots$	$\vdots$	

$$\int u \, dv = uv - \int v \, du$$

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx \qquad \star$$

$\int e^x \sin x \, dx$

	D	I	
u	$\sin x$	e^x	dv
du	$\cos x$	e^x	v

$$\int u \, dv = uv - \int v \, du$$

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$$

$\star$ $$\int e^x \cos x \, dx = e^x \cos x + \underbrace{e^x \sin x - \int e^x \cos x \, dx}$$

$$2 \int e^x \cos x \, dx = e^x \cos x + e^x \sin x + C_1$$

$$\int e^x \cos x \, dx = \frac{1}{2}\left[e^x \cos x + e^x \sin x\right] + C$$

Integral	u	dv
$\int x^n e^x\,dx$	x^n	e^x
$\int x^n \cos x\,dx$	x^n	$\cos x$
$\int x^n \sqrt{1+x}\,dx$	x^n	$\sqrt{1+x}$
$\int x^n \ln x\,dx$	$\ln x$	x^n
$\int x^n \tan^{-1} x\,dx$	$\tan^{-1} x$	x^n