28.6 Inverse Trig Forms

$$\boxed{\begin{aligned} \int \frac{du}{\sqrt{a^2-u^2}} &= \sin^{-1}\left(\frac{u}{a}\right)+C \\ \int \frac{du}{a^2+u^2} &= \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C \end{aligned}}$$

Ex: a) $\int \frac{dx}{\sqrt{36-x^2}}$

$$= \int \frac{dx}{\sqrt{6^2-x^2}}$$

$$= \sin^{-1}\left(\frac{x}{6}\right)+C$$

b) $\int \frac{dx}{\sqrt{36-7x^2}}$

$$= \int \frac{dx}{\sqrt{6^2-(\sqrt{7}x)^2}}$$

$$\boxed{\begin{aligned} u &= \sqrt{7}x \\ du &= \sqrt{7}\,dx \\ dx &= \frac{du}{\sqrt{7}} \end{aligned}}$$

$$= \frac{1}{\sqrt{7}}\int \frac{du}{\sqrt{6^2-u^2}}$$

$$= \frac{1}{\sqrt{7}}\sin^{-1}\left(\frac{u}{6}\right)+C$$

$$= \frac{1}{\sqrt{7}}\sin^{-1}\left(\frac{\sqrt{7}x}{6}\right)+C$$

c) $$\int \frac{x\,dx}{\sqrt{36-16x^4}}$$

$$= \int \frac{x\,dx}{\sqrt{6^2-(4x^2)^2}}$$

$u = 4x^2$
$du = 8x\,dx$
$x\,dx = \frac{du}{8}$

$$= \frac{1}{8}\int \frac{du}{\sqrt{6^2-u^2}}$$

$$= \frac{1}{8}\sin^{-1}\left(\frac{u}{6}\right) + C$$

$$= \frac{1}{8}\sin^{-1}\left(\frac{4x^2}{6}\right) + C$$

or $\frac{1}{8}\sin^{-1}\left(\frac{2x^2}{3}\right) + C$

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d) $$\int_{0.1}^{0.2} \frac{e^x\,dx}{\sqrt{9-e^{2x}}}$$

$$= \int_{0.1}^{0.2} \frac{e^x\,dx}{\sqrt{3^2-(e^x)^2}}$$

$u = e^x$
$du = e^x\,dx$
When $x = 0.1$ $u = e^{0.1}$
$x = 0.2$ $u = e^{0.2}$

$$= \int_{e^{0.1}}^{e^{0.2}} \frac{du}{\sqrt{3^2-u^2}}$$

$$= \left[\sin^{-1}\left(\frac{u}{3}\right) \right]_{e^{0.1}}^{e^{0.2}}$$

$$= \sin^{-1}\left(\frac{e^{0.2}}{3}\right) - \sin^{-1}\left(\frac{e^{0.1}}{3}\right)$$

$$\approx 0.04$$

$\boxed{\text{Radian Mode}}$

<u>Ex</u>: a) $$\int_0^4 \frac{1}{5+x^2}\,dx$$

$$= \int_0^4 \frac{1}{\sqrt{5}^2 + x^2}\,dx$$

$$= \left[\frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{x}{\sqrt{5}}\right) \right]_0^4$$

$$= \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{4}{\sqrt{5}}\right) - \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{0}{\sqrt{5}}\right)$$

$\tan^{-1} 0 = 0$

$$= \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{4}{\sqrt{5}}\right)$$

$$\approx 0.47$$

$\boxed{\text{Radian Mode}}$

b) $$\int \frac{\cos x \, dx}{9+\sin^2 x}$$

$$\boxed{\begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array}}$$

$$= \int \frac{u}{3^2+u^2}$$

$$= \frac{1}{3} \tan^{-1} \frac{u}{3} + C$$

$$= \frac{1}{3} \tan^{-1} \left(\frac{\sin x}{3}\right) + c$$

c) $$\int \frac{x \, dx}{9+3x^4}$$

$$= \int \frac{x \, dx}{3^2 + (\sqrt{3}x^2)^2}$$

$$\boxed{\begin{array}{c} u = \sqrt{3}\, x^2 \\ du = 2\sqrt{3}\, x \, dx \\ x \, dx = \dfrac{du}{2\sqrt{3}} \end{array}}$$

$$= \frac{1}{2\sqrt{3}} \int \frac{du}{3^2+u^2}$$

$$= \frac{1}{2\sqrt{3}} \left[\frac{1}{3} \tan^{-1}\left(\frac{u}{3}\right) \right] + c$$

$$= \frac{1}{6\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{3}x^2}{3}\right) + C$$

d) $\displaystyle\int \frac{e^{-t}\,dt}{1+9e^{-2t}}$

$\displaystyle = \int \frac{e^{-t}\,dt}{1+(3e^{-t})^2}$

$u = 3e^{-t}$

$du = -3e^{-t}\,dt$

$e^{-t}\,dt = \dfrac{du}{-3}$

$\displaystyle = \frac{-1}{3}\int \frac{du}{1+u^2}$

$= -\frac{1}{3}\tan^{-1} u + C$

$= -\frac{1}{3}\tan^{-1}(3e^{-t}) + C$

e) $\displaystyle\int \frac{dx}{x^2+6x+20}$ $\leftarrow$

Complete the square

$x^2+\boxed{6}x+20 = (x+3)^2 + \,?$

$\frac{6}{2}=3$

$= (x+3)^2 + 11$

$= (x+3)^2 + \sqrt{11}^2$

$\displaystyle = \int \frac{dx}{(x+3)^2+\sqrt{11}^2}$

$u = x+3$

$du = dx$

$\displaystyle = \int \frac{du}{\sqrt{11}^2+u^2}$

$= \frac{1}{\sqrt{11}}\tan^{-1}\left(\frac{u}{\sqrt{11}}\right) + C$

$= \frac{1}{\sqrt{11}}\tan^{-1}\left(\frac{x+3}{\sqrt{11}}\right) + C$

f) $\displaystyle\int \frac{dx}{x^2+8x+20}$

$$\boxed{\begin{aligned} x^2+8x+20 &= (x+4)^2+\ ? \\ &= (x+4)^2+4 \\ &= (x+4)^2+2^2 \end{aligned}}$$

$= \displaystyle\int \frac{dx}{(x+4)^2+2^2}$

$$\boxed{\begin{aligned} u &= x+4 \\ du &= dx \end{aligned}}$$

$= \displaystyle\int \frac{du}{2^2+u^2}$

$= \frac{1}{2}\tan^{-1}\frac{u}{2} + c$

$= \frac{1}{2}\tan^{-1}\left(\frac{x+4}{2}\right) + c$

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g) $\displaystyle\int \frac{dx}{\sqrt{x}(1+x)}$

$= \displaystyle\int \frac{1}{1+\sqrt{x}^2} \cdot \frac{dx}{\sqrt{x}}$

$$\boxed{\begin{aligned} u &= \sqrt{x} \\ du &= \tfrac{1}{2}x^{-1/2}\,dx \\ du &= \frac{dx}{2\sqrt{x}} \\ \frac{dx}{\sqrt{x}} &= 2du \end{aligned}}$$

$= 2\displaystyle\int \frac{1}{1+u^2}\,du$

$= 2\tan^{-1}u + c$

$= 2\tan^{-1}\sqrt{x} + c$

RECAP

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C \qquad \text{ARCSINE}$$

$$\int \frac{du}{1+u^2} = \tan^{-1} u + C \qquad \text{ARCTAN}$$

$$\int \frac{du}{u} = \ln|u| + C \qquad \text{LN}$$

$$\int u^n \, du = \frac{u^{n+1}}{n+1} + C \qquad \text{POWER}$$

Ex: Identify the form: arcsine, arctan, ln, power, none.

a) $\int \frac{dx}{1+x^2}$ ARCTAN

b) $\int \frac{x\,dx}{1+x^2}$ LN

$= \frac{1}{2}\int \frac{du}{u}$

c) $\int \frac{x\,dx}{\sqrt{1+x^2}}$ POWER

$= \frac{1}{2}\int \frac{du}{\sqrt{u}}$

d) $\int \frac{x\,dx}{\sqrt{1-x^2}}$ POWER

$= -\frac{1}{2}\int \frac{du}{\sqrt{u}}$

e) $\int \frac{dx}{\sqrt{1-x^2}}$ ARCSINE

f) $\int \frac{dx}{1-x}$ LN

$= -\int \frac{du}{u}$

g) $\int \frac{dx}{1+x}$ LN

$= \int \frac{du}{u}$

h) $\int \frac{dx}{\sqrt{x^2-1}}$ NONE

i) $\int \frac{dx}{\sqrt{1+x^2}}$ NONE

j) $\int \frac{dx}{1-x^2}$ NONE