1. [5 marks] Find an explicit solution (solve for y):

$dy - x^3e^{4x}dx - \frac{3}{x}ydx = 0$

Linear

1) Standard form

$dy - \frac{3}{x}ydx = x^3e^{4x}dx$

2) I.F.

$P(x) = -\frac{3}{x}$

$\int P(x)dx = -3\ln x$

$e^{\int P(x)dx} = e^{-3\ln x} = e^{\ln x^{-3}} = x^{-3}$ (1)

3)

$x^{-3}dy - 3x^{-4}ydx = e^{4x}dx$

$d(x^{-3}y) = e^{4x}dx$

4) Integrate

$x^{-3}y = \frac{e^{4x}}{4} + C$ (2)

Solve for y:

Mult by x^3: $y = \frac{x^3e^{4x}}{4} + Cx^3$ (1)

2. [5 marks] Find an implicit solution (you don't need to solve for y):

$\frac{dy}{dx} - \frac{y^2}{\sec 3x} = 0$ if $y = 3$ when $x = \frac{\pi}{6}$

Separable

$$\frac{dy}{dx} = \frac{y^2}{\sec 3x}$$

$$\frac{dy}{y^2} = \frac{dx}{\sec 3x}$$

(1)

$$\frac{1}{\sec 3x} = \cos 3x$$

$$\int \frac{dy}{y^2} = \int \cos 3x \, dx$$

$$-y^{-1} = \frac{\sin 3x}{3} + C$$

(3)

Sub $y=3$

$x = \frac{\pi}{6}$:

$$-\frac{1}{3} = \frac{1}{3}\sin\frac{\pi}{2} + C$$

$$-\frac{2}{3} = C$$

$$-y^{-1} = \frac{1}{3}\sin 3x - \frac{2}{3}$$

(1)

or $\frac{1}{y} + \frac{\sin 3x}{3} = \frac{2}{3}$ etc.

3. [2 marks] Write down a DE that describes the following statement:
The rate of change of V with respect to r is proportional to r^2

$$\frac{dV}{dr} = kr^2$$

4. [3 marks] Solve $y''' - 3y'' - 40y' = 0$

$$m^3 - 3m^2 - 40m = 0$$
$$m(m^2 - 3m - 40) = 0$$
$$m(m-8)(m+5) = 0$$
$$m = 0, 8, -5$$
$$y = C_1 e^{0x} + C_2 e^{8x} + C_3 e^{-5x}$$
or $$y = C_1 + C_2 e^{8x} + C_3 e^{-5x}$$

5. [2 marks] Solve $y'' + 18y' + 81y = 0$

$$m^2 + 18m + 81 = 0$$

$$(m+9)(m+9) = 0$$

$$m = -9, -9$$

$$y = C_1 e^{-9x} + C_2 x e^{-9x}$$

or $$y = (C_1 + C_2 x) e^{-9x}$$

6. [3 marks] Solve $y'' - 4y' + 13y = 0$

$$m^2 - 4m + 13 = 0$$

$$m = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 13}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$\sqrt{-36} = \sqrt{36}\sqrt{-1} = 6j$$

$$= \frac{4 \pm 6j}{2}$$

$$= 2 \pm 3j$$

$$y = e^{2x}(C_1 \sin 3x + C_2 \cos 3x)$$

7. [5 marks] Solve $y'' - y' - 2y = 3e^{6x} + 7x$ given that $y_C = C_1e^{-x} + C_2e^{2x}$

1) $y_c = C_1e^{-x} + C_2e^{2x}$

2) $y_p = Ae^{6x} + Bx + C$ (1)

$y_p' = 6Ae^{6x} + B$

$y_p'' = 36Ae^{6x}$

3) $y_p \to DE$

$y'' - y' - 2y = 3e^{6x} + 7x$

$36Ae^{6x} - (6Ae^{6x} + B) - 2(Ae^{6x} + Bx + C) = 3e^{6x} + 7x$

$36Ae^{6x} - 6Ae^{6x} - B - 2Ae^{6x} - 2Bx - 2C = 3e^{6x} + 7x$

$28Ae^{6x} - 2Bx - B - 2C = 3e^{6x} + 7x + 0$

$28A = 3$

$A = \frac{3}{28}$

$-2B = 7$

$B = \frac{-7}{2}$

$-B - 2C = 0$

$\frac{7}{2} - 2C = 0$

$2C = \frac{7}{2}$

$C = \frac{7}{4}$ (3)

$y_p = \frac{3}{28}e^{6x} - \frac{7}{2}x + \frac{7}{4}$ (1)

4) $y = y_c + y_p$

$y = C_1e^{-x} + C_2e^{2x} + \frac{3}{28}e^{6x} - \frac{7}{2}x + \frac{7}{4}$