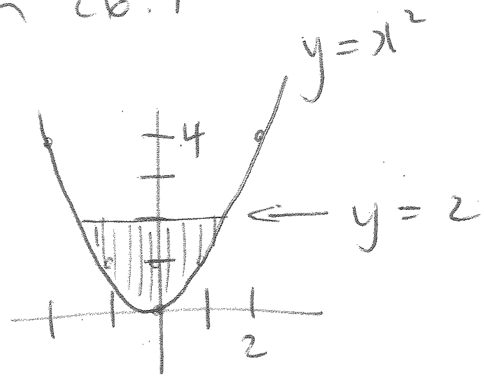
Section 26.4

(11)



$\bar{x}=0$ by symmetry.

Intersection: $y=y$

$x^2=2$

$x=\pm\sqrt{2}$

$$A=\int_{-\sqrt{2}}^{\sqrt{2}}(y_t-y_b)\,dx$$

$$=\int_{-\sqrt{2}}^{\sqrt{2}}(2-x^2)\,dx$$

$$=\left[2x-\frac{x^3}{3}\right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$=2\sqrt{2}-\frac{2\sqrt{2}}{3}-\left(-2\sqrt{2}+\frac{2\sqrt{2}}{3}\right)$$

$$=\frac{4\sqrt{2}}{3}-\left(-\frac{4\sqrt{2}}{3}\right)$$

$$=\frac{8\sqrt{2}}{3}$$

(11) Cont'd

A vertical slice has
$$y_e = \frac{y_b + y_t}{2}$$
$$= \frac{x^2+2}{2}$$
$$= \frac{x^2}{2} + 1$$

$$\int_A y_e\,dA = \int_{-\sqrt{2}}^{\sqrt{2}} \left(\frac{x^2}{2}+1\right)\left(2-x^2\right)dx$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left(x^2 - \frac{x^4}{2} + 2 - x^2\right)dx$$

$$= \left[\frac{-x^5}{10} + 2x\right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= -\frac{4\sqrt{2}}{10} + 2\sqrt{2} - \left(\frac{4\sqrt{2}}{10} - 2\sqrt{2}\right)$$

$$= 4\sqrt{2} - \frac{8\sqrt{2}}{10}$$

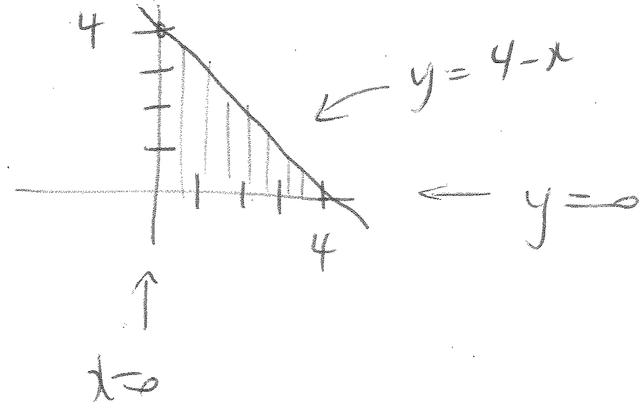
$$= \frac{32\sqrt{2}}{10}$$

$$= \frac{16\sqrt{2}}{5}$$

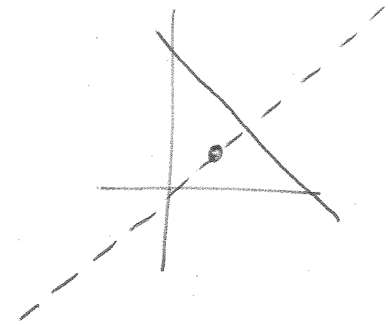
$$\bar{y} = \frac{1}{A}\int_A y_e\,dA = \left(\frac{16\sqrt{2}}{5}\right)\left(\frac{3}{8\sqrt{2}}\right) = \frac{6}{5}$$

$$\Rightarrow (\bar{x}, \bar{y}) = \left(0, \frac{6}{5}\right)$$

(13)



By symmetry, $\bar{y} = \bar{x}$



Centroid will lie on the line $y = x$ by symmetry.

$\boxed{\text{Find } \bar{x}}$

$$A = \int_0^4 (y_t - y_b)\,dx$$
$$= \int_0^4 (4 - x)\,dx$$
$$= \left[4x - \frac{x^2}{2} \right]_0^4$$
$$= 8$$

A vertical slice has $x_e = x$.

$$\int_A x_e\,dA = \int_0^4 x(4 - x)\,dx$$

→

(13) Cont'd

$$= \int_0^4 (4x - x^2)\,dx$$

$$= \left[2x^2 - \frac{x^3}{3} \right]_0^4$$

$$= 32 - \frac{64}{3}$$

$$= \frac{32}{3}$$

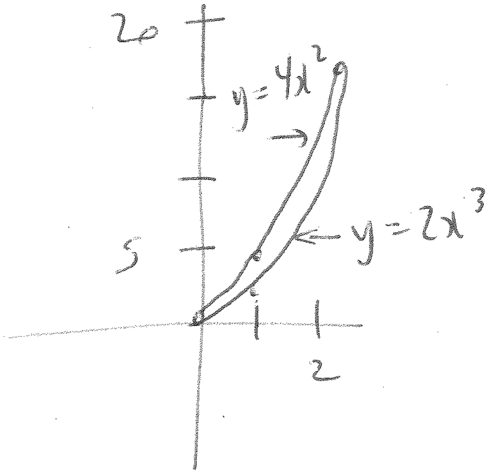
$$\bar{x} = \frac{1}{A} \int_A x_e \, dA$$

$$= \left(\frac{32}{3}\right)\left(\frac{1}{8}\right)$$

$$= \frac{4}{3}$$

$$\Rightarrow (\bar{x}, \bar{y}) = \left(\frac{4}{3}, \frac{4}{3}\right)$$

(15)



Intersection

$$y = y$$
$$4x^2 = 2x^3$$

$$4x^2 - 2x^3 = 0$$
$$2x^2(2 - x) = 0$$
$$x = 0, 2$$

$$A = \int_0^2 (y_t - y_b)\,dx$$
$$= \int_0^2 (4x^2 - 2x^3)\,dx$$
$$= \left[\frac{4x^3}{3} - \frac{x^4}{2}\right]_0^2$$
$$= \frac{32}{3} - 8$$
$$= \frac{8}{3}$$

For a vertical slice,

$$x_e = x$$
$$y_e = \frac{y_b + y_t}{2}$$
$$= \frac{2x^3 + 4x^2}{2}$$
$$= x^3 + 2x^2$$

$\longrightarrow$

(15) Cont'd

$$\int_A x_e \, dA$$

$$= \int_0^2 x(4x^2 - 2x^3)\,dx$$

$$= \int_0^2 (4x^3 - 2x^4)\,dx$$

$$= \left[x^4 - \frac{2x^5}{5}\right]_0^2$$

$$= 16 - \frac{64}{5}$$

$$= \frac{16}{5}$$

$$\int_A y_e \, dA$$

$$= \int_0^2 (x^3 + 2x^2)(4x^2 - 2x^3)\,dx$$

$$= \int_0^2 (4x^5 - 2x^6 + 8x^4 - 4x^5)\,dx$$

$$= \left[-\frac{2x^7}{7} + \frac{8x^5}{5}\right]_0^2$$

$$= -\frac{256}{7} + \frac{256}{5}$$

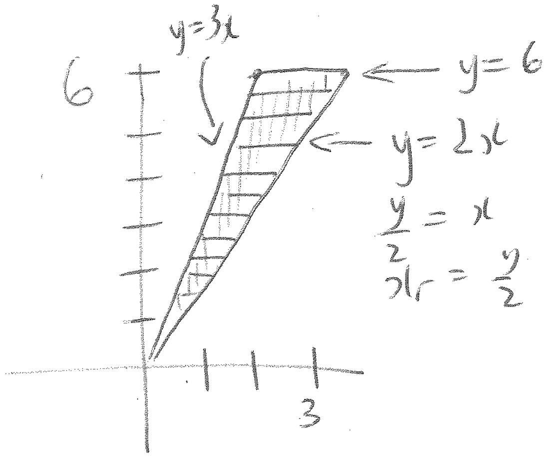
$$= \frac{512}{35}$$

$$\bar{x} = \frac{1}{A}\int_A x_e \, dA = \frac{16}{5}\left(\frac{3}{8}\right) = \frac{6}{5}$$

$$\bar{y} = \frac{1}{A}\int_A y_e \, dA = \frac{512}{35}\left(\frac{3}{8}\right) = \frac{192}{35}$$

or $(\bar{x}, \bar{y}) \approx (1.20, 5.49)$

(17)



$\frac{y}{2} = x$

$x_r = \frac{y}{2}$

$$\boxed{\begin{aligned} y &= 3x \\ x &= \frac{y}{3} \\ x_l &= \frac{y}{3} \end{aligned}}$$

$$A = \int_0^6 (x_r - x_l)\,dy$$

$$= \int_0^6 \left[\frac{y}{2} - \frac{y}{3}\right]dy$$

$$= \left[\frac{y^2}{4} - \frac{y^2}{6}\right]_0^6$$

$$= 3$$

For a horizontal slice:

$$x_e = \frac{x_l + x_r}{2}$$

$$= \frac{1}{2}\left(\frac{y}{3} + \frac{y}{2}\right)$$

$$= \frac{1}{2}\left(\frac{5y}{6}\right)$$

$$= \frac{5y}{12}$$

$$y_e = y$$

$$\int_A x_e\,dA = \int_0^6 \frac{5y}{12}\left(\frac{y}{2} - \frac{y}{3}\right)dy \quad \rightarrow$$

17 Cont'd

$$= \int_0^6 \frac{5y}{12}\left(\frac{y}{6}\right) dy$$

$$= \frac{5}{72}\int_0^6 y^2\, dy$$

$$= \frac{5}{72}\left[\frac{y^3}{3}\right]_0^6$$

$$= 5$$

$$\int_A y\, e\, dA = \int_0^6 y\left(\frac{y}{2}-\frac{y}{3}\right) dy$$

$$= \int_0^6 y \cdot \frac{y}{6}\, dy$$

$$= \int_0^6 \frac{y^2}{6}\, dy$$

$$= \left[\frac{y^3}{18}\right]_0^6$$

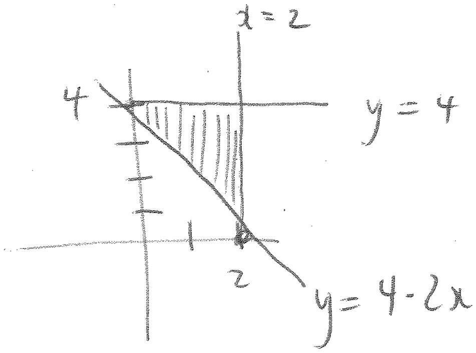
$$= 12$$

$$\bar{x} = \frac{1}{A}\int_A x\, e\, dA = 5\left(\frac{1}{3}\right) = \frac{5}{3}$$

$$\bar{y} = \frac{1}{A}\int_A y\, e\, dA = 12\left(\frac{1}{3}\right) = 4$$

$$(\bar{x}, \bar{y}) = \left(\frac{5}{3}, 4\right)$$

(19)



$$A = \int_0^2 [4-(4-2x)]\,dx$$

$$= \int_0^2 2x\,dx \leftarrow dA$$

$$= [x^2]_0^2$$

$$= 4$$

$x_e = x$ $y_e = \dfrac{y_b + y_t}{2}$

Vertical Slice

$$= \frac{4-2x+4}{2}$$

$$= 4 - x$$

$$\int_A x_e\,dA = \int_0^2 x(2x)\,dx$$

$$= \int_0^2 2x^2\,dx$$

$$= \left[\frac{2x^3}{3}\right]_0^2$$

$$= \frac{16}{3} \rightarrow$$

(19) Cont'd

$$\int_A y_e \, dA = \int_0^2 (4-x)(2x)\,dx$$

$$= \int_0^2 (8x - 2x^2)\,dx$$

$$= \left[4x^2 - \frac{2x^3}{3}\right]_0^2$$

$$= 16 - \frac{16}{3}$$

$$= \frac{32}{3}$$

$$\bar{x} = \frac{1}{A}\int_A x_e \, dA$$

$$= \frac{1}{4}\left(\frac{16}{3}\right)$$

$$= \frac{4}{3}$$

$$\bar{y} = \frac{1}{A}\int_A y_e \, dA$$

$$= \frac{1}{4}\left(\frac{32}{3}\right)$$

$$= \frac{8}{3}$$

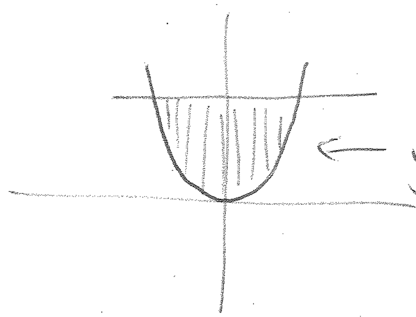
$$\Rightarrow (\bar{x}, \bar{y}) = \left(\frac{4}{3}, \frac{8}{3}\right)$$

(21) The curves are $x^2 = 4py$ and $y = a$.

$$y = \frac{x^2}{4p}$$

For simplicity, let $\frac{1}{4p} = b^2$ and $a = c^2$.

Now the curves are $y = b^2x^2$ and $y = c^2$.



$\boxed{\text{By symmetry, } \bar{x} = 0}$

Intersection:
$$y = y$$
$$b^2x^2 = c^2$$
$$x^2 = \frac{c^2}{b^2}$$
$$x = \pm \frac{c}{b}$$

$$A = \int_{-\frac{c}{b}}^{\frac{c}{b}} (c^2 - b^2x^2)\,dx$$

$$= \left[c^2x - \frac{b^2x^3}{3} \right]_{-\frac{c}{b}}^{\frac{c}{b}}$$

$$= \frac{c^3}{b} - \frac{c^3}{3b} - \left(\frac{-c^3}{b} + \frac{c^3}{3b} \right)$$

$$= \frac{2c^3}{3b} + \frac{2c^3}{3b} \rightarrow$$

(21) Cont'd $= \dfrac{4c^3}{3b}$

For a vertical slice, $y_e = \dfrac{y_b + y_t}{2}$

$$= \frac{b^2x^2 + c^2}{2}$$

$$= \frac{b^2x^2}{2} + \frac{c^2}{2}$$

$$\int_A y_e \, dA = \int_{-\frac{c}{b}}^{\frac{c}{b}} \left(\frac{b^2x^2}{2} + \frac{c^2}{2}\right)\left(c^2 - b^2x^2\right) dx$$

$$= \int_{-\frac{c}{b}}^{\frac{c}{b}} \left[\frac{b^2c^2x^2}{2} - \frac{b^4x^4}{2} + \frac{c^4}{2} - \frac{b^2c^2x^2}{2} \right] dx$$

$$= \left[-\frac{b^4x^5}{10} + \frac{c^4x}{2} \right]_{-\frac{c}{b}}^{\frac{c}{b}}$$

$$= \left[\frac{-c^5}{10b} + \frac{c^5}{2b} \right] - \left[\frac{c^5}{10b} - \frac{c^5}{2b} \right]$$

$$= \frac{4c^5}{10b} + \frac{4c^5}{10b}$$

$$= \frac{4c^5}{5b}$$

$$\bar{y} = \frac{1}{A}\int_A y_e \, dA = \left(\frac{4c^5}{5b}\right)\left(\frac{3b}{4c^3}\right) = \frac{3c^2}{5} = \frac{3}{5}a$$

Note: The answer does not depend on b or p.

$$(\bar{x}, \bar{y}) = \left(0, \frac{3a}{5}\right)$$