# Math 187

## Series problems

1. Find the sum of the geometric series
   (a) $3 + \frac{3}{5} + \frac{3}{25} + \frac{3}{125} + \cdots$
   (b) $\frac{2}{3} - \frac{4}{9} + \frac{8}{27} - \cdots$

2. Find the first two nonzero terms of the Maclaurin series of $\tan x$.

3. Find the first four nonzero terms of the Maclaurin series of
   (a) $x^2 e^x$
   (b) $x^3 \sin(2x)$
   (c) $\sin x \cos x$
   (d) $e^{-x} \cos x$
   (e) $(1 + x)e^{2x}$

4. Explain why it is impossible to find a Maclaurin series expansion for $\ln x$, $\csc x$, and $\sqrt{x}$.

5. Use the geometric series
$$\frac{1}{1-x} = 1 + x + x^2 + \cdots, \quad -1 < x < 1$$
   to find the Maclaurin series of $\ln(1 + x)$ and $\tan^{-1} x$.

6. Use three terms of an appropriate series to approximate the following integrals.
   (a) $\displaystyle\int_0^1 \cos x^2 \, dx$
   (b) $\displaystyle\int_0^1 \sqrt{1 + x^3} \, dx$
   (c) $\displaystyle\int_0^1 e^{-x^2} dx$
   (d) $\displaystyle\int_{0.2}^{0.5} \frac{\sin x}{x} dx$

7. Use Maclaurin series to approximate the following.
   (a) $\sin 3°$ (using two terms)
   (b) $e^{-0.1}$ (using three terms)
   (c) $\ln(1.2)$ (using three terms)
   (d) $\sqrt[3]{1.03}$ (using two terms)

8. Find three terms of the Taylor series of $\sin x$ around $\pi/3$ and use it to estimate $\sin 62°$.

9. Find three terms of the Taylor series of $\sqrt{x}$ in powers of $x - 4$ and use it to estimate $\sqrt{4.3}$.

10. Find three terms of the Taylor series of $\tan x$ centered at $a = \frac{\pi}{4}$ and use it to estimate $\tan 44°$.

11. Use a linear approximation (i.e., two terms of a Taylor series) of $\sqrt[3]{x}$ around $x = 8$ to estimate $\sqrt[3]{8.03}$.

**Answers**

1. (a) 15/4 (b) 2/5

2. $\tan x \approx x + \frac{1}{3}x^3$

3. (a) $x^2 e^x = x^2 + x^3 + \frac{1}{2}x^4 + \frac{1}{6}x^5 + \cdots$
   (b) $x^3 \sin(2x) = 2x^4 - \frac{4}{3}x^6 + \frac{4}{15}x^8 - \frac{8}{315}x^{10} + \cdots$
   (c) $\sin x \cos x = x - \frac{2}{3}x^3 + \frac{2}{15}x^5 - \frac{4}{315}x^7 + \cdots$
   (d) $e^{-x}\cos x = 1 - x + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \cdots$
   (e) $(1+x)e^{2x} = 1 + 3x + 4x^2 + \frac{10}{3}x^3 + \cdots$

4. The Maclaurin series of a function $f(x)$ is given by

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

   For $f(x)$ to have a Maclaurin series, it is necessary that $f^{(n)}(0)$ exists for all $n = 0, 1, 2, \ldots$ Both functions $\ln x$ and $\csc x$ are not defined at $x = 0$ so they can't have a Maclaurin series. For the function $f(x) = \sqrt{x}$, we have $f'(x) = \frac{1}{2\sqrt{x}}$, then $f'(0)$ does not exist so it can't have a Maclaurin series.

5. From the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots, \quad -1 < x < 1$$

   we get

$$\ln(1+x) = \int \frac{dx}{1+x} = \int (1 - x + x^2 - x^3 + \cdots)\, dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

   and

$$\tan^{-1} x = \int \frac{dx}{1+x^2} = \int (1 - x^2 + x^4 - x^6 + \cdots)\, dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

   Note that we set both constants of integration to zero since $\ln(1+0) = 0$ and $\tan^{-1} 0 = 0$.

6. (a) $\int_0^1 \cos x^2\, dx \approx 0.9046$ (b) $\int_0^1 \sqrt{1+x^3}\, dx \approx 1.1071$
   (c) $\int_0^1 e^{-x^2} dx \approx 0.767$ (d) $\int_{0.2}^{0.5} \frac{\sin x}{x} dx \approx 0.29355$

7. (a) $\sin 3^\circ \approx 0.05233$ (b) $e^{-0.1} \approx 0.905$ (c) $\ln(1.2) \approx 0.18267$ (d) $\sqrt[3]{1.03} \approx 1.01$

8. $\sin x \approx \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{3}\right)^2$ and $\sin 62^\circ \approx 0.88295$.

9. $\sqrt{x} \approx 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$ and $\sqrt{4.3} \approx 2.07359$.

10. $\tan x \approx 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2$ and $\tan 44^\circ \approx 0.9657$.

11. $\sqrt[3]{x} \approx 2 + \frac{1}{12}(x-8)$ for all $x$ close to 8. If we set $x = 8.03$, we get $\sqrt[3]{8.03} \approx 2 + \frac{1}{12}(0.03) = 2.0025$.