(1) $\int \frac{3}{x^2-25}\,dx$

$= \int \frac{3}{(x+5)(x-5)}\,dx$

$$\frac{3}{(x+5)(x-5)} = \frac{A}{x+5} + \frac{B}{x-5}$$

$$3 = A(x-5) + B(x+5)$$

sub $x=5$: $3 = 10B$ $\qquad B = 3/10$

sub $x=-5$: $3 = -10A$ $\qquad A = -3/10$

$$\int \left(\frac{-3}{10} \cdot \frac{1}{x+5} + \frac{3}{10} \cdot \frac{1}{x-5}\right) dx$$

$= \frac{-3}{10} \ln|x+5| + \frac{3}{10} \ln|x-5| + C$

or $\frac{3}{10}\left(-\ln|x+5| + \ln|x-5|\right) + C$

$= \frac{3}{10}\left(\ln\left|\frac{1}{|x+5|}\right| + \ln|x-5|\right) + C$

$= \frac{3}{10} \ln\left|\frac{x-5}{x+5}\right| + C$

(2) $\int \frac{\sqrt{x^2-25}}{x}\,dx$

Sub $x = 5\sec\theta$

$dx = 5\sec\theta\tan\theta\,d\theta$

$\sqrt{x^2-25} = 5\tan\theta$

$$\int \frac{5\tan\theta \cdot 5\sec\theta\tan\theta\,d\theta}{5\sec\theta}$$

$= \int 5\tan^2\theta\,d\theta$

$= \int 5(\sec^2\theta - 1)\,d\theta$

$= 5(\tan\theta - \theta) + C$

$= 5\tan\theta - 5\theta + C$

$= \sqrt{x^2-25} - 5\cos^{-1}\left(\frac{5}{x}\right) + C$

or $\sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right) + C$

$5\tan\theta = \sqrt{x^2-25}$

$\theta = ?$

$\frac{x}{5} = \sec\theta$

$\theta = \sec^{-1}\left(\frac{x}{5}\right)$

or $\frac{5}{x} = \cos\theta$

$\theta = \cos^{-1}\left(\frac{5}{x}\right)$

③ $\int \frac{\sqrt{36+x^2}}{x}\,dx$

Sub $x = 6\tan\theta$

$dx = 6\sec^2\theta\, d\theta$

$\sqrt{36+x^2} = 6\sec\theta$

$\int \frac{6\sec\theta \cdot \cancel{6}\sec^2\theta\, d\theta}{\cancel{6}\tan\theta}$

$\sec^2\theta$

$= \int \frac{6\cdot \sec\theta\,(1+\tan^2\theta)}{\tan\theta}\, d\theta$

$= \int \frac{6\sec\theta}{\tan\theta}\, d\theta + \int 6\sec\theta\tan\theta\, d\theta$

$= \int 6\csc\theta\, d\theta + \int 6\sec\theta\tan\theta\, d\theta$

$\frac{\sec\theta}{\tan\theta} = \frac{1}{\cos\theta}\cdot\left(\frac{\cos\theta}{\sin\theta}\right)$

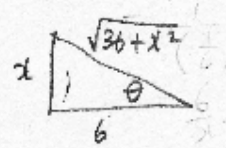
$= \csc\theta$.

$= 6\ln|\csc\theta - \cot\theta| + 6\sec\theta + C$

$= 6\ln\left|\frac{\sqrt{36+x^2}}{x} - \frac{6}{x}\right| + \sqrt{36+x^2} + C$

$6\sec\theta = \sqrt{36+x^2}$

Using $\frac{x}{6} = \tan\theta$:



$\csc\theta = \frac{H}{O} = \frac{\sqrt{36+x^2}}{x}$

$\cot\theta = \frac{A}{O} = \frac{6}{x}$

④ $\int \frac{y^3+1}{y^3+5y^2+4y}\,dy \rightarrow$

$$y^3+5y^2+4y \overline{\big)\, y^3+0y^2+0y+1}$$ quotient 1

$$\frac{-\;y^3+5y^2+4y}{-5y^2-4y+1}$$

$= \int \left(1 + \frac{-5y^2-4y+1}{y^3+5y^2+4y}\right) dy$

$= \int \left(1 + \frac{-5y^2-4y+1}{y(y^2+5y+4)}\right) dy$

$= \int \left(1 + \frac{-5y^2-4y+1}{y(y+1)(y+4)}\right) dy$

$\rightarrow$

$$\frac{-5y^2-4y+1}{y(y+1)(y+4)} = \frac{A}{y} + \frac{B}{y+1} + \frac{C}{y+4}$$

$$-5y^2-4y+1 = A(y+1)(y+4) + By(y+4) + Cy(y+1)$$

Sub $y=0$: $\quad 1 = 4A \qquad A = 1/4$

$y=-1$: $\quad 0 = -3B \qquad B = 0$

$y=-4$: $\quad -63 = 12C \qquad C = \frac{-63}{12} = \frac{-21}{4}$

Integral $= \int \left(1 + \frac{1}{4}\cdot\frac{1}{y} - \frac{21}{4}\cdot\frac{1}{y+4}\right) dy$

$= y + \frac{1}{4}\ln|y| - \frac{21}{4}\ln|y+4|$

or $\quad y + \frac{1}{4}\ln\left|\frac{y}{(y+4)^{21}}\right| + C$

(5) $\displaystyle\int \frac{1-4x^2}{x^2(x+4)}\,dx$

repeated linear factor

$$\frac{1-4x^2}{x^2(x+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+4}$$

$$1-4x^2 = Ax(x+4) + B(x+4) + Cx^2$$

Sub $x=0$ $\quad 1 = 4B \quad \boxed{B = 1/4}$

Sub $x=-4$ $\quad -63 = 16C \quad \boxed{C = -63/16}$

Compare coefficients :

Coefficient of x^2

$[x^2]$ $\quad -4 = A + C$

$-4 = A - \frac{63}{16}$

$-4 + \frac{63}{16} = A$

$\boxed{A = -1/16}$

$$\text{Integral} = \int\left(-\frac{1}{16}\cdot\frac{1}{x} + \frac{1}{4}\cdot\frac{1}{x^2} - \frac{63}{16}\cdot\frac{1}{x+4}\right)dx$$

$$= -\frac{1}{16}\ln|x| - \frac{1}{4}x^{-1} - \frac{63}{16}\ln|x+4| + C$$

or $\quad -\frac{1}{16}\ln\left|x(x+4)^{63}\right| - \frac{1}{4}x^{-1} + C$

(6) $\int \frac{2x^2+3}{(x-1)(x^2+4)}\,dx$

quadratic factor

$$\frac{2x^2+3}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$$

$$2x^2+3 = A(x^2+4) + (Bx+C)(x-1)$$

Sub $x=1$: $5 = 5A$ $\boxed{A=1}$

$[x^2]$ ← Coefficient of x^2

$$2 = A+B$$
$$2 = 1+B$$
$$\boxed{B=1}$$

$[x^0]$ ← Constants

$$3 = 4A-C$$
$$3 = 4-C$$
$$\boxed{C=1}$$

$$\text{Integral} = \int\left(\frac{1}{x-1} + \frac{x+1}{x^2+4}\right)dx$$
$$= \int\left(\frac{1}{x-1} + \frac{x}{x^2+4} + \frac{1}{x^2+4}\right)dx$$
$$= \int\left(\frac{1}{x-1} + \frac{1}{2}\cdot\frac{2x}{x^2+4} + \frac{1}{x^2+4}\right)dx$$
$$= \ln|x-1| + \frac{1}{2}\ln|x^2+4| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$$

or $\ln\left|(x-1)\sqrt{x^2+4}\right| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$

(7) $f(x,y) = e^{x}\cos y + e^{-2x}\tan y$

$$\frac{\partial f}{\partial x} = e^{x}\cos y - 2e^{-2x}\tan y$$

$$\frac{\partial f}{\partial y} = -e^{x}\sin y + e^{-2x}\sec^2 y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(-e^{x}\sin y + e^{-2x}\sec^2 y\right)$$

$$= -e^{x}\sin y - 2e^{-2x}\sec^2 y$$

(8) a) $(r, \theta, z) = (9, \frac{\pi}{3}, 5)$

$x = r\cos\theta$ $\quad y = r\sin\theta$

$= 9\cos\frac{\pi}{3}$ $\quad = 9\sin\frac{\pi}{3}$

$= \frac{9}{2}$ $\quad = \frac{9\sqrt{3}}{2}$

$(x, y, z) = (\frac{9}{2}, \frac{9\sqrt{3}}{2}, 5)$

b) $(x, y, z) = (-4, 4\sqrt{3}, 1)$

$r = \sqrt{x^2 + y^2}$

$= \sqrt{(-4)^2 + (4\sqrt{3})^2}$

$= \sqrt{16 + 48}$

$= \sqrt{64}$

$= 8$

$\tan\theta = \frac{y}{x}$

$\tan\theta = -\sqrt{3}$

$x<0 \quad y>0$

$\theta = \tan^{-1}(-\sqrt{3}) \quad (+\pi?)$

$\theta = -\frac{\pi}{3} + \pi$

$\theta = \frac{2\pi}{3}$

$(r, \theta, z) = (8, \frac{2\pi}{3}, 1)$

(9)

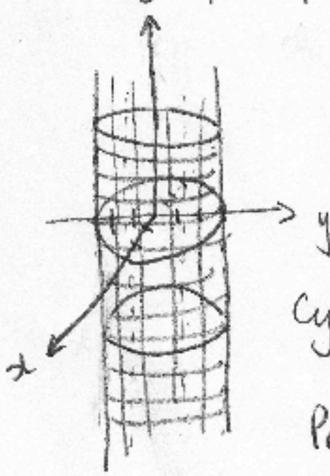
$$3x^2 + 3y^2 = 27$$

$$3(r\cos\theta)^2 + 3(r\sin\theta)^2 = 27$$

$$3r^2\cos^2\theta + 3r^2\sin^2\theta = 27$$

$$3r^2 = 27$$

$$r^2 = 9$$



Circle of radius 3
Centred @ origin $r^2 = 9$
for every z-value

Cylinder
of radius 3
Parallel to z-axis

(10) $$\int_1^2 \int_x^{x^2} x^2 y \, dy \, dx$$

$$= \int_1^2 \left[x^2 \frac{y^2}{2} \right]_x^{x^2} dx$$

$$= \int_1^2 \left[x^2 \frac{(x^2)^2}{2} - x^2 \frac{x^2}{2} \right] dx$$

$$= \int_1^2 \left(\frac{x^6}{2} - \frac{x^4}{2} \right) dx$$

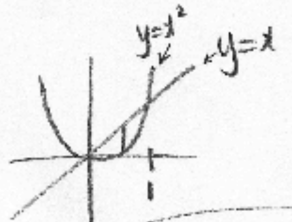
$$= \left[\frac{x^7}{14} - \frac{x^5}{10} \right]_1^2$$

$$= \left(\frac{128}{14} - \frac{32}{10} \right) - \left(\frac{1}{14} - \frac{1}{10} \right)$$

$$= \frac{836}{140}$$

or $\frac{209}{35}$

(11) First octant : $x, y, z > 0$



$$\boxed{\begin{array}{c} x^2 \le y \le x \\ 0 \le x \le 1 \end{array}}$$

$$\text{Volume} = \int_0^1 \int_{x^2}^{x} z \, dy \, dx$$

$$= \int_0^1 \int_{x^2}^{x} xy \, dy \, dx$$

$$= \int_0^1 \left[\frac{xy^2}{2}\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[\frac{x(x)^2}{2} - \frac{x \cdot (x^2)^2}{2}\right] dx$$

$$= \int_0^1 \left(\frac{x^3}{2} - \frac{x^5}{2}\right) dx$$

$$= \left[\frac{x^4}{8} - \frac{x^6}{12}\right]_0^1$$

$$= \left[\frac{1}{8} - \frac{1}{12}\right] - [0]$$

$$= \frac{1}{24}$$