① a) $AB = \begin{bmatrix} 1 & -4 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} -14 & -18 \\ -22 & -26 \end{bmatrix}$

$BA = \begin{bmatrix} -2 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 13 & -42 \end{bmatrix}$

b) $AB = \begin{bmatrix} 1 & -1 & 2 \\ 4 & -3 & 3 \end{bmatrix} \begin{bmatrix} -2 & 5 \\ 1 & 6 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 8 \end{bmatrix}$

$BA = \begin{bmatrix} -2 & 5 \\ 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 4 & -3 & 3 \end{bmatrix} = \begin{bmatrix} 18 & -13 & 11 \\ 25 & -19 & 20 \\ 13 & -11 & 16 \end{bmatrix}$

c) $AB = \begin{bmatrix} 1 & -1 & 2 \\ 4 & -3 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 13 \\ -8 \end{bmatrix} = \begin{bmatrix} -20 \\ -27 \end{bmatrix}$

BA is undefined

B   A

3×1 2×3

must be equal

(2) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(provided $ad-bc \neq 0$)

a) $ad-bc = 1(-7)-(-6)(4) = 17$

$A^{-1} = \frac{1}{17}\begin{bmatrix} -7 & 6 \\ -4 & 1 \end{bmatrix}$

b) $ad-bc = 12(2)-(-3)(8) = 0$

B^{-1} does not exist

(3)

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ R_2-2R_1 \\ R_3-2R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 0 & 1 & 0 \\ 0 & -5 & -1 & 1 & -2 & 0 \\ 0 & -4 & 0 & 0 & -2 & 1 \end{array}\right]$$

$$R_2/(-5) \quad \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1/5 & -1/5 & 2/5 & 0 \\ 0 & -4 & 0 & 0 & -2 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1-3R_2 \\ \\ R_3+4R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 2/5 & 3/5 & -1/5 & 0 \\ 0 & 1 & 1/5 & -1/5 & 2/5 & 0 \\ 0 & 0 & 4/5 & -4/5 & -\frac{2}{5} & 1 \end{array}\right]$$

$$\begin{array}{l} \\ \\ R_3 \times \frac{5}{4} \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 2/5 & 3/5 & -1/5 & 0 \\ 0 & 1 & 1/5 & -1/5 & 2/5 & 0 \\ 0 & 0 & 1 & -1 & -\frac{1}{2} & 5/4 \end{array}\right]$$

$$\begin{array}{l} R_1-2/5R_3 \\ R_2-1/5R_3 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 1 & -1 & -1/2 & 5/4 \end{array}\right] \leftarrow A^{-1} = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1/2 & -1/4 \\ -1 & -1/2 & 5/4 \end{bmatrix}$$

(4) a) $\begin{bmatrix} 8 & -1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 17 \\ -5 \end{bmatrix}$ $(AX=B)$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 & -1 \\ -4 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 17 \\ -5 \end{bmatrix} \quad (X=A^{-1}B)$$

$$\begin{bmatrix} 8 & -1 \\ -4 & 1 \end{bmatrix}^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 4 & 8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} 17 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 12 \\ 28 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

b) To solve $AX=B$, use $X=A^{-1}B$

To find A^{-1}:

$$\left[\begin{array}{ccc|ccc} 4 & 0 & 4 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 4 & 0 & 4 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ R_2-4R_1 \\ R_3-R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & -4 & -4 & 1 & -4 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right]$$

$$R_2/(-4) \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1/4 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right]$$

$$R_1-R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/4 & 0 & 0 \\ 0 & 1 & 1 & -1/4 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right]$$

$$R_3/(-1) \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/4 & 0 & 0 \\ 0 & 1 & 1 & -1/4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{array}{l} R_1-R_3 \\ R_2-R_3 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & -1 & 1 \\ 0 & 1 & 0 & -1/4 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$\uparrow$

A^{-1}

See next page →

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/4 & -1 & 1 \\ -1/4 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} -4 \\ -4 \\ -2 \end{bmatrix} \qquad (X = A^{-1}B)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

5 a)

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 9 \\ 2 & -1 & 4 & 6 \\ -3 & 2 & -3 & -1 \end{array}\right]$$

$$\begin{array}{l} \\ R_2-2R_1 \\ R_3+3R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 9 \\ 0 & -7 & 8 & -12 \\ 0 & 11 & -9 & 26 \end{array}\right]$$

$$\begin{array}{l} \\ R_2/(-7) \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 9 \\ 0 & 1 & -8/7 & 12/7 \\ 0 & 11 & -9 & 26 \end{array}\right]$$

$$\begin{array}{l} R_1-3R_2 \\ \\ R_3-11R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 10/7 & 27/7 \\ 0 & 1 & -8/7 & 12/7 \\ 0 & 0 & 25/7 & 50/7 \end{array}\right]$$

$$\begin{array}{l} \\ \\ R_3 \times \frac{7}{25} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 10/7 & 27/7 \\ 0 & 1 & -8/7 & 12/7 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{array}{l} R_1-\frac{10}{7}R_3 \\ R_2+\frac{8}{7}R_3 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{aligned} x &= 1 \\ y &= 4 \\ z &= 2 \end{aligned} \qquad (x,y,z) = (1,4,2)$$

b) $\left[\begin{array}{ccc|c} 3 & -18 & 21 & 12 \\ 2 & 7 & -6 & 3 \end{array}\right]$

$R_1/3$ $\left[\begin{array}{ccc|c} 1 & -6 & 7 & 4 \\ 2 & 7 & -6 & 3 \end{array}\right]$

R_2-2R_1 $\left[\begin{array}{ccc|c} 1 & -6 & 7 & 4 \\ 0 & 19 & -20 & -5 \end{array}\right]$

$R_2/19$ $\left[\begin{array}{ccc|c} 1 & -6 & 7 & 4 \\ 0 & 1 & -20/19 & -5/19 \end{array}\right]$

R_1+6R_2 $\left[\begin{array}{ccc|c} 1 & 0 & 13/19 & 46/19 \\ 0 & 1 & -20/19 & -5/19 \end{array}\right]$ $x + \frac{13}{19}z = \frac{46}{19}$, $y - \frac{20}{19}z = \frac{-5}{19}$

So $x = \frac{46}{19} - \frac{13}{19}z$, $y = \frac{-5}{19} + \frac{20}{19}z$

z = any real number

Can also be written

$(x,y,z) = \left(\frac{46}{19} - \frac{13}{19}t, \frac{-5}{19} + \frac{20}{19}t, t\right)$

any real #

Two particular solutions:

$(z=0)$ $(x,y,z) = \left(\frac{46}{19}, \frac{-5}{19}, 0\right)$ $(z=1)$ $\left(\frac{33}{19}, \frac{15}{19}, 1\right)$

c) $\left[\begin{array}{ccc|c} 1 & 3 & 3 & 12 \\ 2 & 20 & 10 & 8 \\ 1 & 10 & 5 & 0 \end{array}\right]$

$\begin{array}{c} \\ R_2-2R_1 \\ R_3-R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 12 \\ 0 & 14 & 4 & -16 \\ 0 & 7 & 2 & -12 \end{array}\right]$

$R_2/14 \left[\begin{array}{ccc|c} 1 & 3 & 3 & 12 \\ 0 & 1 & 2/7 & -8/7 \\ 0 & 7 & 2 & -12 \end{array}\right]$

$\begin{array}{c} R_1-3R_2 \\ \\ R_3-7R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 15/7 & 108/7 \\ 0 & 1 & 2/7 & -8/7 \\ 0 & 0 & 0 & -4 \end{array}\right]$ ← impossible

$0x + 0y + 0z = -4$
is impossible

No solution
"System is inconsistent."