3.3 Geometric Sequences and Series

A <u>geometric sequence</u> is a sequence in which the next term is the previous term multiplied by a constant.

The constant is called the <u>common ratio</u>, written r.

<u>Ex</u>: Find r in the geometric sequences below:

a) $7, 14, 28, \ldots$

$$r = \frac{14}{7} = 2$$

b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{256}$

$$r = \frac{\left(\frac{1}{4}\right)}{\left(\frac{1}{2}\right)} = \frac{1}{4} \times \frac{2}{1} = \frac{1}{2}$$

c) $24, -8, \frac{8}{3}, \frac{-8}{9}, \ldots$

$$r = \frac{-8}{24} = \frac{-1}{3}$$

Recursive Formula for Infinite Geometric Sequences

$$\begin{cases} a_m = \text{<insert first term here>} \\ a_n = r a_{n-1} & \text{for } n \geq m+1 \end{cases}$$

<u>Ex</u>: Give a recursive formula for
$100, 20, 4, \frac{4}{5}, \ldots$

Geometric Sequence with $r = \frac{1}{5}$

$$\begin{cases} a_1 = 100 \\ a_n = \frac{1}{5} a_{n-1} & \text{for } n \geq 2 \end{cases}$$

General Formula for Infinite Geometric Sequences

$$a_n = a_m r^{n-m} \qquad \text{for } n \geq m$$

<u>Ex</u>: Find a general formula for
$7, 14, 28, \ldots$

Geometric Sequence with $r = 2$

$$a_n = a_m r^{n-m} \qquad \text{for } n \geq m$$

Let $m = 1$: $$a_n = a_1 \cdot r^{n-1} \qquad \text{for } n \geq 1$$

$$a_n = 7 \cdot 2^{n-1} \qquad \text{for } n \geq 1$$

Ex: Find a general formula for

$80, -160, 320, \ldots$

$\boxed{\text{Geometric Sequence with } r=-2}$

$a_n = a_m r^{n-m}$ for $n \geq m$

Let $m=1$: $a_n = a_1 r^{n-1}$ for $n \geq 1$

$a_n = 80(-2)^{n-1}$ for $n \geq 1$

Ex: Consider $5, 15, 45, \ldots$

Find the twelfth term.

$\boxed{\text{Geometric Sequence with } r=3}$

$a_n = a_m r^{n-m}$ for $n \geq m$

Let $m=1$: $a_n = a_1 r^{n-1}$ for $n \geq 1$

$a_n = 5 \cdot 3^{n-1}$ for $n \geq 1$

$a_{12} = 5 \cdot 3^{11}$

$= 885735$

Geometric series: A sum in which the next term is the previous term multiplied by r.

A geometric series: $7+14+28+\ldots$

Fact

$$\boxed{S_k = \frac{a_m(1-r^k)}{1-r}}$$

where k = # terms in partial sum

a_m = first term

r = common ratio

Ex: Consider $2+10+50+\ldots$

Find the sum of the first 12 terms.

Geometric series with $r=5$

$$S_k = \frac{a_m(1-r^k)}{1-r}$$

$$S_{12} = \frac{2(1-5^{12})}{(1-5)}$$

$$= \frac{-488281248}{-4}$$

$$= 122070312$$

Fact

$$\boxed{\begin{array}{l}\text{If } -1<r<1 \\ \text{then } r^k \to 0 \text{ as } k\to\infty\end{array}}$$

Quick Ex: $0.9^{100} \approx 0$ ✓

$(-0.9)^{100} \approx 0$ ✓

Fact

$$\boxed{\begin{array}{l}\text{If } -1<r<1 \text{ then } S_\infty = \dfrac{a_m}{1-r} \\ \text{If } r\geq 1 \text{ or } r\leq -1 \text{ then } S_\infty \text{ is undefined.}\end{array}}$$

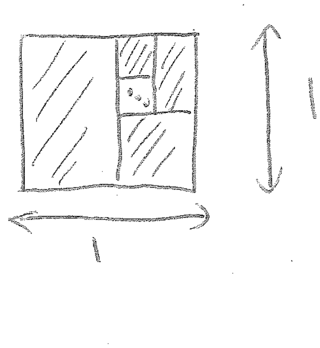
Why? $S_k = \dfrac{a_m(1-r^k)}{1-r}$ $\boxed{r^k \text{ goes to } 0 \text{ as } k\to\infty}$

Ex: Calculate $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots$

Geometric Series with $r=\frac{1}{2}$

$$S_\infty = \frac{a_m}{1-r} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)} = 1$$

Visual Proof:



Ex: Calculate $24 - 16 + \frac{32}{3} - \dots$

Geometric series with $r = \frac{-16}{24} = \frac{-2}{3}$

$$S_\infty = \frac{a_m}{1-r}$$
$$= \frac{24}{\left(1+\frac{2}{3}\right)}$$
$$= \frac{24}{\left(\frac{5}{3}\right)}$$
$$= 24 \times \frac{3}{5}$$
$$= \frac{72}{5} \text{ or } 14.4$$

Ex: Calculate $12 + 18 + 27 + \dots$

Geometric series with $r = \frac{18}{12} = \frac{3}{2}$

S_∞ is undefined.

Ex: Calculate $\sum_{j=0}^{\infty} 75\left(\frac{3}{5}\right)^j = 75 + 45 + \dots$

Geometric series with $r = \frac{45}{75} = \frac{3}{5}$

$$S_\infty = \frac{a_m}{1-r}$$
$$= \frac{75}{\left(\frac{2}{5}\right)}$$
$$= 75 \times \frac{5}{2}$$
$$= 187.5$$

Ex: Calculate $\sum_{j=0}^{10} 75\left(\frac{3}{5}\right)^j$

Geometric series with $r = \frac{3}{5}$

$$S_k = \frac{a_m(1-r^k)}{1-r}$$

$$\boxed{\begin{aligned} k &= \#\text{ terms} \\ &= n-m+1 \\ &= 10-0+1 \\ &= 11 \end{aligned}}$$

$$S_{11} = \frac{75\left(1-\left(\frac{3}{5}\right)^{11}\right)}{\left(1-\frac{3}{5}\right)}$$

$$\approx 186.82$$