Binary, Logic, and More

Applied Math for Computing

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Chapter 1

Binary, Octal, and Hexadecimal

1.1 Decimal and Octal

1.1.1 Review of the Decimal System

Before we look at the numbering systems commonly used by computers, it will likely be helpful to review the workings of the decimal system, the numbering system commonly used by humans. The decimal system (base 10) is based on ten digits, starting from zero, and uses a positional notation, so called because the magnitude of the number depends not only on what digits are used, but also **where** each digit is located within the number.

For example, if we start counting from zero upwards, we get

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$$

Notice that in base ten, we don't have a single digit to denote the number ten. Instead, we write a zero in the right column and then write a one in the column to the left. Similarly, when we continue counting up to twenty,

$$11, 12, 13, 14, 15, 16, 17, 18, 19, 20$$

once we have written the number nineteen as 19, the next number sets the right digit to zero while incrementing the left column by one to give the number 20 (twenty).

This means that for the the decimal number 179, the digit 9 is in the "ones" place, the digit 7 is in the "tens" place, and the digit 1 is in the "hundreds" place, so we can write

$$179 = 1 \times 100 + 7 \times 10 + 9 \times 1$$

or

$$179 = 1 \times 10^2 + 7 \times 10^1 + 9 \times 10^0,$$

recalling that $10^0 = 1$.

Example: In the decimal number 386, state which digit is in the

(a) ones place

(b) tens place

(c) hundreds place

Answer:

(a) 6, since it's the right-most digit

(b) 8

(c) 3

Example: In the decimal number 24680, in what place are the following digits?

(a) 8

(b) 6

(c) 0

(d) 2

(e) 4

Answer:

(a) the tens place

(b) the hundreds place

(c) the ones place

(d) the ten thousands place

(e) the thousands place

1.1.2 Bases Other Than Ten – How They Work

To put a number into a base other than ten, we use the same ideas as before:

- the number of digits available is equal to the base
- there is no single digit which represents the base, so in order to write the base in that system, set the right column to zero and increment the column to the left by one

The best way to understand this is to work through an example, so let us first look at numbers written in base 4. This base is **not** commonly used in computing, but it is a useful example nonetheless. We will use the same ideas as before:

- there are four digits in total: 0,1,2,3
- there is no single digit which represents the base, so when we want to write the number "four", set the right column to zero and increment the column to the left by one

To make it clear which base we are using, any numbers written in a base other than ten will have the base as a subscript. So the number three in base 4 is 3_4.

Let's contrast counting using base 10 versus base 4 by counting from one to twenty in both bases side-by-side. Notice that the default base is 10, so numbers in the decimal system are written without a subscript, but numbers in base 4 have the base as a subscript.

base 10	base 4
1	1_4
2	2_4
3	3_4
4	10_4
5	11_4
6	12_4
7	13_4
8	20_4
9	21_4
10	22_4

base 10	base 4
11	23_4
12	30_4
13	31_4
14	32_4
15	33_4
16	100_4
17	101_4
18	102_4
19	103_4
20	110_4

Another thing to note is what happens when we try to write the decimal number 16 in base 4. The previous number, 15, is written as 33_4. When you add one to 33_4, the three in the right-hand column increments to four, but since that is the base, we write a zero and add one to the next column over. Since that is also a three, we set that column to zero and write a one in the next column, so that $16 = 100_4$.

So, looking at the number 14 in decimal, we can think of it as

$$14 = 1 \times 10 + 4 \times 1$$

but that same number written in base 4 is

$$32_4 = 3 \times 4 + 2 \times 1$$

and since 12 + 2 is 14, you can see that the two numerical representations are equivalent.

In the same way that we expanded 179 earlier as

$$179 = 1 \times 10^2 + 7 \times 10^1 + 9 \times 10^0,$$

we can expand numbers in base 4 in the same way by replacing the base 10 with the base 4. So

$$\begin{aligned} 100_4 &= 1 \times 4^2 + 0 \times 4^1 + 0 \times 4^0 \\ &= 1 \times 16 + 0 \times 4 + 0 \times 1 \\ &= 16 + 0 + 0 \\ &= 16 \end{aligned}$$

and we can conclude that $100_4 = 16_{10}$, as we discussed.

Similarly,

$$\begin{aligned} 302_4 &= 3 \times 4^2 + 0 \times 4^1 + 2 \times 4^0 \\ &= 3 \times 16 + 0 \times 4 + 2 \times 1 \\ &= 48 + 0 + 2 \\ &= 50 \end{aligned}$$

and we can conclude that $302_4 = 50_{10}$.

Example: In the number 132_4, state which digit is in the

(a) ones place

(b) fours place

(c) sixteens place

Answer:

(a) 2

(b) 3

(c) 1

Example: The number 1230_4 can be expanded in base 10 as

$$\begin{aligned} 1230_4 &= 1 \times 4^3 + 2 \times 4^2 + 3 \times 4^1 + 0 \times 4^0 \\ &= 64 + 32 + 12 + 0 \\ &= 108 \end{aligned}$$

Expand the following numbers into base 10 in a similar fashion. Then perform that calculation to find the number when written in decimal form.

(a) 23_4

(b) 121_4

(c) 30102_4

(d) 2132_4

Answer:

(a) $23_4 = 2 \times 4^1 + 3 \times 4^0 = 8 + 3 = 11$

(b) $121_4 = 1 \times 4^2 + 2 \times 4^1 + 1 \times 4^0 = 16 + 8 + 1 = 25$

(c) $30102_4 = 3 \times 4^4 + 0 \times 4^3 + 1 \times 4^2 + 0 \times 4^1 + 2 \times 4^0 = 768 + 0 + 16 + 0 + 2 = 786$

(d) $2132_4 = 2 \times 4^3 + 1 \times 4^2 + 3 \times 4^1 + 2 \times 4^0 = 128 + 16 + 12 + 2 = 158$

1.1.3 Octal

Let us now look at numbers written in base 8, called octal. Octal is a base commonly used in computing. We will use the same ideas as before:

- there are eight digits in total: 0,1,2,3,4,5,6,7
- there is no single digit which represents the base, so when we want to write the number "eight", set the right column to zero and increment the column to the left by one

So, in base 8, we can only count to seven using single digits:

$$0, 1, 2, 3, 4, 5, 6, 7$$

and then the number after that is 10 (in base 8). To make it clear which base we are using, any numbers written in a base other than ten will have the base as a subscript. So the number eight in base 8 is 10_8.

Let's contrast counting using base 10 versus base 8 by counting from one to twenty in both bases side-by-side. Again, the numbers in octal will have the base as a subscript.

base 10	base 8
1	1_8
2	2_8
3	3_8
4	4_8
5	5_8
6	6_8
7	7_8
8	10_8
9	11_8
10	12_8

base 10	base 8
11	13_8
12	14_8
13	15_8
14	16_8
15	17_8
16	20_8
17	21_8
18	22_8
19	23_8
20	24_8

So, looking at the number 14 in decimal, we can think of it as

$$14 = 1 \times 10 + 4 \times 1$$

but that same number written in octal is

$$16_8 = 1 \times 8 + 6 \times 1$$

and since 8 + 6 is 14, you can see that the two numerical representations are equivalent.

In the same way that we expanded 179 earlier as

$$179 = 1 \times 10^2 + 7 \times 10^1 + 9 \times 10^0,$$

we can expand numbers in octal in the same way by replacing the base 10 with the base 8.

So

$$\begin{aligned} 245_8 &= 2 \times 8^2 + 4 \times 8^1 + 5 \times 8^0 \\ &= 2 \times 64 + 4 \times 8 + 5 \times 1 \\ &= 128 + 32 + 5 \\ &= 165 \end{aligned}$$

and we can conclude that $245_8 = 165_{10}$.

Example: In the number 135724_8, state which digit is in the

(a) ones place

(b) eights place

(c) sixty-fours place

(d) 8^5 place

Answer:

(a) 4

(b) 2

(c) 7

(d) 1

Example: The number 12345_8 can be expanded in base 10 as $1 \times 8^4 + 2 \times 8^3 + 3 \times 8^2 + 4 \times 8^1 + 5 \times 8^0$. Expand the following numbers into base 10 in a similar fashion. Then perform that calculation to find the number when written in decimal form.

(a) 41_8

(b) 764_8

(c) 1011_8

(d) 25073_8

Answer:

(a) $41_8 = 4 \times 8^1 + 1 \times 8^0 = 32 + 1 = 33$

(b) $764_8 = 7 \times 8^2 + 6 \times 8^1 + 4 \times 8^0 = 448 + 48 + 4 = 500$

(c) $1011_8 = 1 \times 8^3 + 0 \times 8^2 + 1 \times 8^1 + 1 \times 8^0 = 512 + 0 + 8 + 1 = 521$

(d) $25073_8 = 2 \times 8^4 + 5 \times 8^3 + 0 \times 8^2 + 7 \times 8^1 + 3 \times 8^0 = 8192 + 2560 + 0 + 56 + 3 = 10811$

Example: Use the technique of the previous example to expand the following numbers with different bases into base 10. Then perform that calculation to find the number when written in decimal form.

(a) 210_3

(b) 11001_2

(c) 4135_6

(d) 266_7

Answer:

(a) $210_3 = 2 \times 3^2 + 1 \times 3^1 + 0 \times 3^0 = 18 + 3 + 0 = 21$

(b) $11001_2 = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 16 + 8 + 0 + 0 + 1 = 25$

(c) $4135_6 = 4 \times 6^3 + 1 \times 6^2 + 3 \times 6^1 + 5 \times 6^0 = 864 + 36 + 18 + 5 = 923$

(d) $266_7 = 2 \times 7^2 + 6 \times 7^1 + 6 \times 7^0 = 98 + 42 + 6 = 146$

Exercises for Section 1.1

Consider the table below.

base 10	base 2	base 3	base 4	base 5	base 6	base 7	base 8
1			1_4				
2			2_4				
3			3_4				
4			10_4				
5			11_4				
6			12_4				
7			13_4				
8			20_4				
9			21_4				
10			22_4				
11			23_4				
12			30_4				
13							
14							
15							
16							
17							
18							
19							
20							

For the following exercises, complete the specified column in this table. The fourth column has been started as an example.

1. base 2
2. base 3
3. base 4
4. base 5

5. base 6

6. base 7

7. base 8

In the number 12345678_{10}, in what place are the following digits?

8. 8

9. 6

10. 5

11. 7

12. 2

13. 1

In the number 1234567_8, which digit is in the

14. ones place?

15. eights place?

16. sixty-fours place?

17. 8^5 place?

The number 12345_8 can be expanded in base 10 as $1 \times 8^4 + 2 \times 8^3 + 3 \times 8^2 + 4 \times 8^1 + 5 \times 8^0$. Expand the following numbers into base 10 in a similar fashion.

18. 523_8

19. 1011110_2

20. 22013_4

21. 4130_5

22. 987_{10}

Convert the following numbers to base 10:

23. 7231_8
24. 2031_4
25. 100_8
26. 1005_8
27. 2034_8

Answers to Section 1.1 Exercises

Here is the table for questions 1-7:

base 10	base 2	base 3	base 4	base 5	base 6	base 7	base 8
1	1_2	1_3	1_4	1_5	1_6	1_7	1_8
2	10_2	2_3	2_4	2_5	2_6	2_7	2_8
3	11_2	10_3	3_4	3_5	3_6	3_7	3_8
4	100_2	11_3	10_4	4_5	4_6	4_7	4_8
5	101_2	12_3	11_4	10_5	5_6	5_7	5_8
6	110_2	20_3	12_4	11_5	10_6	6_7	6_8
7	111_2	21_3	13_4	12_5	11_6	10_7	7_8
8	1000_2	22_3	20_4	13_5	12_6	11_7	10_8
9	1001_2	100_3	21_4	14_5	13_6	12_7	11_8
10	1010_2	101_3	22_4	20_5	14_6	13_7	12_8
11	1011_2	102_3	23_4	21_5	15_6	14_7	13_8
12	1100_2	110_3	30_4	22_5	20_6	15_7	14_8
13	1101_2	111_3	31_4	23_5	21_6	16_7	15_8
14	1110_2	112_3	32_4	24_5	22_6	20_7	16_8
15	1111_2	120_3	33_4	30_5	23_6	21_7	17_8
16	10000_2	121_3	100_4	31_5	24_6	22_7	20_8
17	10001_2	122_3	101_4	32_5	25_6	23_7	21_8
18	10010_2	200_3	102_4	33_5	30_6	24_7	22_8
19	10011_2	201_3	103_4	34_5	31_6	25_7	23_8
20	10100_2	202_3	110_4	40_5	32_6	26_7	24_8

8. ones
9. hundreds
10. thousands
11. tens

12. millions
13. ten millions
14. 7
15. 6
16. 5
17. 2
18. $523_8 = 5 \times 8^2 + 2 \times 8^1 + 3 \times 8^0$
19. $1011110_2 = 1 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$
20. $22013_4 = 2 \times 4^4 + 2 \times 4^3 + 0 \times 4^2 + 1 \times 4^1 + 3 \times 4^0$
21. $4130_5 = 4 \times 5^3 + 1 \times 5^2 + 3 \times 5^1 + 0 \times 5^0$
22. $987_{10} = 9 \times 10^2 + 8 \times 10^1 + 7 \times 10^0$
23. $7231_8 = 3737$
24. $2031_4 = 141$
25. $100_8 = 64$
26. $1005_8 = 517$
27. $2034_8 = 1052$

1.2 Binary and Hexadecimal

1.2.1 Binary

Let us now look at base 2, called binary. We will use the same ideas as before:

- there are two digits in total: 0,1
- there is no single digit which represents the base, so when we want to write the number "two", set the right column to zero and increment the column to the left by one

So, in base 2, we can only count to one using single digits:

$$0, 1$$

and then the number after that is 10_2. To make it clear which base we are using, any numbers written in a base other than ten will have the base as a subscript. Let's contrast counting using base 10 versus base 2 by counting from one to twenty in both bases side-by-side.

base 10	base 2
1	1_2
2	10_2
3	11_2
4	100_2
5	101_2
6	110_2
7	111_2
8	1000_2
9	1001_2
10	1010_2

base 10	base 2
11	1011_2
12	1100_2
13	1101_2
14	1110_2
15	1111_2
16	10000_2
17	10001_2
18	10010_2
19	10011_2
20	10100_2

So, in order to convert the binary number 110_2 to decimal, we can be expand it as

$$\begin{aligned}110_2 &= 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 \\ &= 1 \times 4 + 1 \times 2 + 0 \times 1 \\ &= 4 + 2 \\ &= 6\end{aligned}$$

and so we can conclude that $110_2 = 6_{10}$. Similarly, the number 10100_2 can be expanded as

$$\begin{aligned}10100_2 &= 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 \\ &= 1 \times 16 + 0 \times 8 + 1 \times 4 + 0 \times 2 + 0 \times 1 \\ &= 16 + 4 \\ &= 20\end{aligned}$$

and $10100_2 = 20_{10}$.

You can see that numbers in binary don't have to be very large before they get quite difficult to read. That is why we generally write numbers in a computing context in bases that are powers of two, like octal, rather than in binary, even though computers themselves only use ones and zeros. We will see in Section 1.5 how to quickly convert back and forth between binary, octal, and hexadecimal.

Example: In the following binary numbers, in what place is the underlined number?

(a) $11100\underline{1}$

(b) $1110\underline{0}1$

(c) $11\underline{1}001$

(d) $\underline{1}11001$

Answer:

(a) the ones place

(b) the twos place

(c) the eights place (the 2^3s place)

(d) the thirty-twos place (the 2^5s place)

Example: Convert the following numbers to base 10.

(a) 11_2

(b) 11010_2

(c) 1010101_2

Answer:

(a) $$\begin{aligned} 11_2 &= 1 \times 2^1 + 1 \times 2^0 \\ &= 1 \times 2 + 1 \times 1 \\ &= 2 + 1 \\ &= 3 \end{aligned}$$

(b) $$\begin{aligned} 11010_2 &= 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 \\ &= 1 \times 16 + 1 \times 8 + 0 \times 4 + 1 \times 2 + 0 \times 1 \\ &= 16 + 8 + 0 + 2 + 0 \\ &= 26 \end{aligned}$$

(c) $$\begin{aligned} 1010101_2 &= 1 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\ &= 1 \times 64 + 0 \times 32 + 1 \times 16 + 0 \times 8 + 1 \times 4 + 0 \times 2 + 1 \times 1 \\ &= 64 + 0 + 16 + 0 + 4 + 0 + 1 \\ &= 85 \end{aligned}$$

1.2.2 Hexadecimal

Another common base used in computing is base 16, called hexadecimal. We will use the same ideas as before:

- there are sixteen digits in total
- there is no single digit which represents the base, so when we want to write the number “sixteen”, set the right column to zero and increment the column to the left by one

The problem arises that in base 10, we have ten digits to use, but we need another six in order to count in hexadecimal. Rather than going with new symbols that might be hard to remember, we use some familiar ones in a very recognizable order:

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F$$

and then the number after that, corresponding to sixteen, is 10_{16}.

Let's contrast counting using base 10 versus base 16 by counting from one to twenty in both bases side-by-side.

base 10	base 16
1	1_{16}
2	2_{16}
3	3_{16}
4	4_{16}
5	5_{16}
6	6_{16}
7	7_{16}
8	8_{16}
9	9_{16}
10	A_{16}

base 10	base 16
11	B_{16}
12	C_{16}
13	D_{16}
14	E_{16}
15	F_{16}
16	10_{16}
17	11_{16}
18	12_{16}
19	13_{16}
20	14_{16}

So the number 14_{16} can be expanded as

$$\begin{aligned} 14_{16} &= 1 \times 16^1 + 4 \times 16^0 \\ &= 1 \times 16 + 4 \times 1 \\ &= 16 + 4 \\ &= 20 \end{aligned}$$

and $14_{16} = 20_{10}$.

Example: In the number $13579BDF_{16}$, in what place are the following digits?

(a) 1

(b) D

(c) F

(d) 5

Answer:

(a) the 16^7s place

(b) the sixteens place

(c) the ones place

(d) the 16^5s place

Example: The number $1234E_{16}$ can be expanded in base 10 as $1 \times 16^4 + 2 \times 16^3 + 3 \times 16^2 + 4 \times 16^1 + 14 \times 16^0$ (recall that in hexadecimal, the digit $E_{16} = 14_{10}$). Expand the following numbers into base 10 in a similar fashion.

(a) $A1_{16}$

(b) $BB8_{16}$

(c) $C1D1_{16}$

(d) $1FFFFD_{16}$

Answer:

(a) $A1_{16} = 10 \times 16^1 + 1 \times 16^0$

(b) $BB8_{16} = 11 \times 16^2 + 11 \times 16^1 + 8 \times 16^0$

(c) $C1D1_{16} = 12 \times 16^3 + 1 \times 16^2 + 13 \times 16^1 + 1 \times 16^0$

(d) $1FFFFD_{16} = 1 \times 16^5 + 15 \times 16^4 + 15 \times 16^3 + 15 \times 16^2 + 15 \times 16^1 + 13 \times 16^0$

Example: Convert the following numbers to base 10:

(a) $9F0_{16}$

(b) $DE4CD_{16}$

Answer:

(a)
$$\begin{aligned} 9F0_{16} &= 9 \times 16^2 + 15 \times 16^1 + 0 \times 16^0 \\ &= 2304 + 240 \\ &= 2544 \end{aligned}$$

(b)
$$\begin{aligned} DE4CD_{16} &= 13 \times 16^4 + 14 \times 16^3 + 4 \times 16^2 + 12 \times 16^1 + 13 \times 16^0 \\ &= 851968 + 57344 + 1024 + 192 + 13 \\ &= 910541 \end{aligned}$$

Exercises for Section 1.2

In the following binary numbers, in what place is the underlined number?

1. $1001010\underline{1}1$
2. $10010101\underline{1}$
3. $10\underline{0}101011$
4. $\underline{1}00101011$
5. $1001\underline{0}1011$

The number 11110_2 can be expanded in base 10 as $1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$. Expand the following numbers into base 10 in a similar fashion. Then perform that calculation to convert the number to base 10.

6. 10_2
7. 111_2
8. 1011_2
9. 1110111_2

Convert the following numbers to base 10.

10. 1001_2
11. 10110001_2
12. 10101_2

In the number $1C3D02_{16}$, in what place are the following digits?

13. 2
14. 0
15. D
16. 3
17. C
18. 1

The number 12345_{16} can be expanded in base 10 as $1 \times 16^4 + 2 \times 16^3 + 3 \times 16^2 + 4 \times 16^1 + 5 \times 16^0$. Expand the following numbers into base 10 in a similar fashion. You do not need to do the full calculation.

19. 523_{16}
20. $F2_{16}$
21. $2A013_{16}$
22. $BEAD_{16}$
23. $9C8_{16}$

Convert the following numbers to base 10.

24. $AC882_{16}$
25. 1000_{16}
26. $2CF_{16}$
27. $BB8_{16}$
28. $7AAA01_{16}$
29. $65ABF_{16}$

Answers to Section 1.2 Exercises

1. the twos place
2. the ones place
3. the 64s place (2^6)
4. the 256s place (2^8)
5. the sixteens (2^4) place
6. $$\begin{aligned} 10_2 &= 1 \times 2^1 + 0 \times 2^0 \\ &= 2 + 0 \\ &= 2 \end{aligned}$$
7. $$\begin{aligned} 111_2 &= 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 \\ &= 4 + 2 + 1 \\ &= 7 \end{aligned}$$
8. $$\begin{aligned} 1011_2 &= 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 \\ &= 8 + 0 + 2 + 1 \\ &= 11 \end{aligned}$$
9. $$\begin{aligned} 1110111_2 &= 1 \times 2^6 + 1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 \\ &= 64 + 32 + 16 + 0 + 4 + 2 + 1 \\ &= 119 \end{aligned}$$
10. $1001_2 = 9$
11. $10110001_2 = 177$
12. $10101_2 = 21$
13. ones
14. sixteens
15. 16^2
16. 16^3
17. 16^4
18. 16^5
19. $523_{16} = 5 \times 16^2 + 2 \times 16^1 + 3 \times 16^0$

20. $F2_{16} = 15 \times 16^1 + 2 \times 16^0$
21. $2A013_{16} = 2 \times 16^4 + 10 \times 16^3 + 0 \times 16^2 + 1 \times 16^1 + 3 \times 16^0$
22. $BEAD_{16} = 11 \times 16^3 + 14 \times 16^2 + 10 \times 16^1 + 13 \times 16^0$
23. $9C8_{16} = 9 \times 16^2 + 12 \times 16^1 + 8 \times 16^0$
24. $AC882_{16} = 706690$
25. $1000_{16} = 4096$
26. $2CF_{16} = 719$
27. $BB8_{16} = 3000$
28. $7AAA01_{16} = 8038913$
29. $65ABF_{16} = 416447$

1.3 Converting Non-Integer Numbers to Decimal

1.3.1 Review of the Decimal System for Non-Integers

Let's once again review the the decimal system, but this time we will consider non-integer numbers. Recall that integers are numbers that can be written without a fractional part, like 5, −3, and 0. To write non-integer numbers as a decimal, we again use positional notation, where the fractional part is to the right of the decimal point.

For example, if we consider the base-10 number

$$8.76$$

then the dot is called the decimal point, the digit to the immediate left (8) is in the "ones" place, and the first digit to the right (7) is in the "tenths" place, while the second digit to the right (6) is in the "hundredths" place. So this number is equal to

$$8.76 = 8 + \frac{7}{10} + \frac{6}{100}.$$

Recalling that $\frac{1}{10}$ can be written as 10^{-1}, then we can rewrite this as

$$8.76 = 8 \times 10^0 + 7 \times 10^{-1} + 6 \times 10^{-2},$$

which is another way of saying that

$$8.76 = 8 + 0.7 + 0.06$$

which may seem redundant but this representation will come in handy when looking at non-decimal bases.

Example: In the decimal number 38.6, state which digit is in the

(a) tens place

(b) ones place

(c) tenths place

Answer:

(a) 3

(b) 8, since it is to the left of the decimal point

(c) 6

Example: In the decimal number 2.4608, in what place are the following digits?

(a) 6

(b) 2

(c) 0

(d) 8

(e) 4

Answer:

(a) the hundredths place

(b) the ones place

(c) the thousandths place

(d) the ten thousandths place

(e) the tenths place

1.3.2 Non-Integers in Bases Other Than Ten

When we consider a number such as

$$10.011_2$$

we immediately run into a naming problem. We can no longer call the dot the "decimal point", as this is not a decimal number. For this particular example, we can call the dot the "binary point", and in general, the dot is called the "radix point".

Now, the digits to the left of the binary point make up the integer part of the number as before, and the digits to the right make up the fractional part.

Let's look at a number in base 4, so we're not getting confused with all of the zeros and ones. If we consider

$$20.31_4$$

we see that the two digits to the left of the radix point, 2 and 0, make up the integer part, while the 3 and the 1 make up the fractional part. Using positional notation, the 3 is in the "fourths" place while the 1 is in the "sixteenths" place. We have seen before that $20_4 = 2 \times 4^1 + 0 \times 4^0$, so now

$$20.31_4 = 2 \times 4^1 + 0 \times 4^0 + 3 \times 4^{-1} + 1 \times 4^{-2}$$

or if you prefer

$$20.31_4 = 2 \times 4 + 0 + \frac{3}{4} + \frac{1}{4^2}.$$

Writing these numbers in their decimal equivalents, we see that

$$\begin{aligned} 20.31_4 &= 8 + 0 + 0.75 + 0.0625 \\ &= 8.8125_{10} \end{aligned}$$

Similarly,

$$\begin{aligned} F2.B9_{16} &= 15 \times 16^1 + 2 \times 16^0 + 11 \times 16^{-1} + 9 \times 16^{-2} \\ &= 15 \times 16 + 2 \times 1 + \frac{11}{16} + \frac{9}{16^2} \\ &= 240 + 2 + 0.6875 + 0.035156 \\ &= 242.72265625 \end{aligned}$$

and we can conclude that $F2.B9_{16} = 242.72265625_{10}$.

Example: In the number $30.1C_{16}$, state which digit is in the

(a) sixteens place

(b) sixteenths place

Answer:

(a) 3

(b) 1

Example: The number 1230.1_4 can be expanded in base 10 as

$$\begin{aligned}1230.1_4 &= 1 \times 4^3 + 2 \times 4^2 + 3 \times 4^1 + 0 \times 4^0 + 1 \times 4^{-1} \\ &= 64 + 32 + 12 + 0 + 0.25 \\ &= 108.25\end{aligned}$$

Expand the following numbers in a similar fashion. Then perform that calculation to find the number when written in decimal form. If appropriate, round your answer to three decimal places.

(a) 101.011_2

(b) 12.17_8

(c) $0.FE_{16}$

(d) 2132.43_5

Answer:

(a) $101.011_2 = 1\times2^2+0\times2^1+1\times2^0+0\times2^{-1}+1\times2^{-2}+1\times2^{-3} = 5.375$

(b) $12.17_8 = 1 \times 8^1 + 2 \times 8^0 + 1 \times 8^{-1} + 7 \times 8^{-2} = 10.234$

(c) $0.FE_{16} = 0 \times 16^0 + 15 \times 16^{-1} + 14 \times 16^{-2} = 0.992$

(d) $2132.43_5 = 2\times5^3+1\times5^2+3\times5^1+2\times5^0+4\times5^{-1}+3\times5^{-2} = 292.92$

Exercises for Section 1.3

In the number 123.45678_{10}, in what place are the following digits?

1. 3
2. 6
3. 5
4. 7
5. 2
6. 1

In the number 1234.567_8, which digit is in the

7. ones place?
8. eighths place?
9. eights place?
10. sixty-fourths place?
11. sixty-fours place?

Convert the following numbers to base 10. When appropriate, round to 3 decimal places.

12. 72.31_8
13. 203.1_4
14. 100.111_2
15. 100.5_7
16. $20C4.B7_{16}$

Answers to Section 1.3 Exercises

1. ones
2. thousandths
3. hundredths
4. ten thousandths
5. tens
6. hundreds
7. 4
8. 5
9. 3
10. 6
11. 2
12. 58.391
13. 35.25
14. 4.875
15. 49.714
16. 8388.715

1.4 Converting from Decimal

As we've seen, converting from a different base back to decimal form can be done by expansion:

$$\begin{aligned} 316_8 &= 3 \times 8^2 + 1 \times 8^1 + 6 \times 8^0 \\ &= 3 \times 64 + 1 \times 8 + 6 \times 1 \\ &= 192 + 8 + 6 \\ &= 206 \end{aligned}$$

but how do we go back the other way? In order to do that, we first need to look at some modular arithmetic.

1.4.1 Modular Arithmetic: Finding Quotient and Remainder

Suppose we wish to divide one integer by another. If the second integer doesn't divide into the first evenly, the result is a real number which we can report either in fraction or decimal form. For example, if we divide 13 by 5, we get

$$13 \div 5 = \frac{13}{5} = 2\frac{3}{5} = 2.6$$

However, if for some reason we wish to stay in the land of integers, we could also report the result by saying that 13 divided by 5 equals 2 plus 3 left over. In this example, the 2 is called the **quotient** and the 3 is called the **remainder**, and we can write this calculation in the following form:

$$13 = \underbrace{2}_{\text{quotient}} \times 5 + \underbrace{3}_{\text{remainder}}$$

This sort of calculation is very helpful when doing unit conversions.[1] For example, if we know that a certain length of time is 54 hours long and we

[1] The metric system has reduced the need for this type of calculation, but since we are still stuck with practical units that are not multiples of 10 (time given in days, hours, and minutes, for example), doing this type of conversion is still, alas, necessary.

would prefer to give it in terms of days and hours, then

$$\begin{aligned} 54 \text{ hours} &= \underbrace{\text{days}}_{\text{quotient}} \times 24 + \underbrace{\text{hours}}_{\text{remainder}} \\ &= 2 \times 24 + 6 \end{aligned}$$

so 54 hours = 2 days plus 6 hours.

But how can we find quotients and remainders with a standard scientific calculator?[2] If we take the number 54 and divide it by 24, our calculator will tell us 2.25. There are then two ways to go:

1. Take the integer part of 2.25, which is 2. Then perform the following calculation:

$$\begin{aligned} \text{remainder} &= 54 - \underbrace{2}_{\text{integer part}} \times 24 \\ &= 54 - 48 \\ &= 6 \end{aligned}$$

2. Alternatively, you can take the decimal part of 2.25, which is 0.25. Multiply this number by 24, the number you are dividing by.

$$\begin{aligned} \text{remainder} &= \underbrace{0.25}_{\text{decimal part}} \times 24 \\ &= 6 \end{aligned}$$

Example: Find the quotient and remainder for the following.

(a) $25 \div 4$

(b) $86 \div 3$

(c) $101 \div 12$

(d) $91 \div 8$

[2] If you wanted to this in code, most computer languages have built-in functions trunc() and mod():

```
days = trunc(54,24)
hours = mod(54,24)
```

where trunc() comes from the word truncate, meaning "to shorten something by cutting off the end" and mod() comes from the mathematical word modulus.

Answer:

(a) $25 \div 4 = 6.25$
quotient = integer part of $6.25 = 6$
remainder $= 25 - \underbrace{6}_{\text{integer part}} \times 4 = 1$
or remainder $= \underbrace{0.25}_{\text{decimal part}} \times 4 = 1$

(b) quotient = 28, remainder = 2

(c) quotient = 8, remainder = 5

(d) quotient = 11, remainder = 3

1.4.2 Using Quotient and Remainder to Convert from Decimal Form

Now that we know how to find quotients and remainders, let's look at some examples that convert from a decimal number into octal. Consider the number 316_8. We earlier found that $316_8 = 206_{10}$. Let's now go back the other way, using repeated division and remainders.

Example: Convert the decimal number 206 to octal.

Answer: The procedure is to divide 206 by the base, which in this case is 8, and write down the quotient and remainder. Then divide the quotient by the base, and write down the new quotient and remainder. As you can see below, we first divide 206 by 8 to get a quotient of 25 with remainder 6. If we then divide 25 by 8, we get quotient 3 with remainder 1. We continue doing this until we get a quotient of zero, as in the table below.

	quotient	remainder
$206 \div 8 =$	25	6
$25 \div 8 =$	3	1
$3 \div 8 =$	0	3

Notice that if you write down the remainders in reverse order, you get the octal number 316_8. Nifty!

Let's do some more examples using this same procedure.

Example: Convert the decimal number 41 to binary.

Answer:

	quotient	remainder
$41 \div 2 =$	20	1
$20 \div 2 =$	10	0
$10 \div 2 =$	5	0
$5 \div 2 =$	2	1
$2 \div 2 =$	1	0
$1 \div 2 =$	0	1

and reading the remainders from bottom to top gives $41_{10} = 101001_2$.

Example: Convert the decimal number 24362 to hexadecimal.

Answer:

	quotient	remainder (base 10)	remainder (base 16)
$24362 \div 16 =$	1522	10	A
$1522 \div 16 =$	95	2	2
$95 \div 16 =$	5	15	F
$5 \div 16 =$	0	5	5

and $24362_{10} = 5F2A_{16}$.

1.4.3 Converting Non-integer Numbers from Decimal Form

We have just seen that to convert an integer number from decimal form, we divide by the base repeatedly. To convert the fractional part of a non-integer decimal number to another base, we instead will multiply by the base repeated.

Example: Convert the decimal number 0.59375 to octal.

Answer: The procedure is to multiply the decimal number by the base, and split the result into the integer part and the fractional (decimal) part. Then take the fractional/decimal part and write it on the next line. Multiply it by 8, and split as before. Once the fractional part goes to zero, you can stop. Finally, write down the integer parts from top to bottom after the radix point and add the subscript for the base.

	integer	fractional
$0.59375 \times 8 =$	4	$+ 0.75$
$0.75 \times 8 =$	6	$+ 0$

and reading the integer parts from top to bottom gives $0.59375_{10} = 0.46_8$.

Example: Convert the decimal number 0.625 to binary.

Answer:

	integer	fractional
$0.625 \times 2 =$	1	$+ 0.25$
$0.25 \times 2 =$	0	$+ 0.5$
$0.5 \times 2 =$	1	$+ 0$

and reading the integer parts from top to bottom gives $0.625_{10} = 0.101_2$.

Example: Convert the decimal number 0.6328125 to hexadecimal.

Answer:

	integer	fractional
$0.6328125 \times 16 =$	10 (A)	$+ 0.125$
$0.125 \times 16 =$	2	$+ 0$

and reading the integer parts from top to bottom gives $0.6328125_{10} = 0.A2_{16}$.

Let's do an example with a twist.

Example: Convert the decimal number 0.3 to octal.

Answer:

	integer	fractional
$0.3 \times 8 =$	2	$+ 0.4$
$0.4 \times 8 =$	3	$+ 0.2$
$0.2 \times 8 =$	1	$+ 0.6$
$0.6 \times 8 =$	4	$+ 0.8$
$0.8 \times 8 =$	6	$+ 0.4$

You'll notice that we have a bit of a problem: if we put the fractional/decimal part 0.4 on the next line, then we'll have a repeat of the second line, and then the next three lines will also repeat, and so on. What this means is that 0.3_{10} is a repeating decimal in octal: $0.3_{10} = 0.23146314631463146... = 0.2\overline{3146}_8$.

Finally, what if we are converting a non-integer number that has both an integer part and a fractional part? The answer is to do the two parts separately and then put them together.

Example: Convert the decimal number 17.375 to binary.

Answer: First, we'll do the integer part and convert 17_{10} to binary.

	quotient	remainder
$17 \div 2 =$	8	1
$8 \div 2 =$	4	0
$4 \div 2 =$	2	0
$2 \div 2 =$	1	0
$1 \div 2 =$	0	1

So $17_{10} = 10001_2$.

Now, convert 0.375_{10} to binary.

	integer	fractional
$0.375 \times 2 =$	0	$+ 0.75$
$0.75 \times 2 =$	1	$+ 0.5$
$0.5 \times 2 =$	1	$+ 0$

So $0.375_{10} = 0.011_2$.

Putting it all together, we get that $17.375_{10} = 10001.011_2$.

Exercises for Section 1.4

Convert the following decimal numbers to the indicated base.

1. 23 to octal
2. 12 to binary
3. 48 to hexadecimal

Convert the decimal number 1234 to the following bases.

4. binary
5. octal
6. hexadecimal
7. base 7

Convert the following decimal numbers to the indicated base.

8. 7203 to octal
9. 123 to binary
10. 11331 to hexadecimal

Perform the following conversions for non-integer numbers. Give exact answers (do not round off).

11. 0.359375 to octal
12. 0.8125 to binary
13. 0.234375 to hexadecimal

Perform the following conversions for non-integer numbers. Use the repeater bar in your answer.

14. 0.6 to octal
15. 0.3 to binary
16. 0.36 to hexadecimal

Perform the following conversions. Give exact answers (do not round off).

17. 18.125 to hexadecimal
18. 31.6 to base 4

19. 37.875 to octal
20. 23.35 to binary

Answers to Section 1.4 Exercises

1. $23 = 27_8$
2. $12 = 1100_2$
3. $48 = 30_{16}$
4. $1234 = 10011010010_2$
5. $1234 = 2322_8$
6. $1234 = 4D2_{16}$
7. $1234 = 3412_7$
8. $7203 = 16043_8$
9. $123 = 1111011_2$
10. $11331 = 2C43_{16}$
11. 0.27_8
12. 0.1101_2
13. $0.3C_{16}$
14. $0.\overline{4631}_8$
15. $0.0\overline{1001}_2$ (if you don't notice the repeating pattern immediately, the answers $0.01\overline{0011}_2$, $0.010\overline{0110}_2$, etc., are also acceptable)
16. $0.\overline{5C28F}_{16}$
17. 12.2_{16}
18. $133.\overline{21}_4$
19. 45.7_8
20. $10111.01\overline{0110}_2$

1.5 Converting between Binary, Octal, and Hexadecimal

1.5.1 Converting Between Binary and Octal

Let's first count from zero to seven in both octal and binary.

octal	binary
0_8	0_2
1_8	1_2
2_8	10_2
3_8	11_2
4_8	100_2
5_8	101_2
6_8	110_2
7_8	111_2

If we add leading zeros where necessary to bring all binary numbers to three digits, we can see that

$$\begin{aligned} 6_8 &= 110_2 \\ 3_8 &= 011_2 \\ 4_8 &= 100_2 \end{aligned}$$

The reason we are doing this is that if we look at a number expressed both in octal and binary, such as $634_8 = 110011100_2$, we can see an interesting pattern. If the binary number is split into groups of three, starting from the right-hand-side,

$$\begin{aligned} 634_8 &= 110011100_2 \\ &= 110\ 011\ 100_2 \\ &= \overbrace{110}^{6}\ \overbrace{011}^{3}\ \overbrace{100}^{4}{}_2 \end{aligned}$$

then we can see that each group of three digits is equal to the corresponding octal number in that place. Nifty, no?

Let's use this observation to convert from octal to binary.

Example: Convert the following octal numbers to binary:

(a) 23_8

(b) 671_8

Answer:

(a) So for each digit in the octal number, write the corresponding 3-digit binary number. Then put the groups of three all together and drop any leading zeros.

$$\begin{aligned} 23_8 &= \overbrace{010}^{2}\,\overbrace{011}^{3}{}_2 \\ &= 010\ 011_2 \\ &= 10\ 011_2\text{, dropping the leading zero} \end{aligned}$$

(b) $$\begin{aligned} 671_8 &= \overbrace{110}^{6}\,\overbrace{111}^{7}\,\overbrace{001}^{1}{}_2 \\ &= 110\ 111\ 001_2 \end{aligned}$$

You can also remove the spaces if you wish, but keeping them makes the binary number a little easier to read.

The same technique can be used to convert non-integer octal numbers to binary.

Example: Convert the following octal numbers to binary:

(a) 0.32_8

(b) 16.07_8

Answer:

(a) For each digit in the octal number, write the corresponding 3-digit binary number. Then put the groups of three all together and drop any leading zeros to the left of the binary point and any trailing zeros to the right of the radix point.

$$\begin{aligned} 0.32_8 &= 0.\,\overbrace{011}^{3}\,\overbrace{010}^{2}{}_2 \\ &= 0.011\ 010_2 \\ &= 0.011\ 01_2\text{, dropping the trailing zero} \end{aligned}$$

(b) $16.07_8 = \overbrace{001}^{1}\ \overbrace{110}^{6}\ .\ \overbrace{000}^{0}\ \overbrace{111}^{7}{}_2$
$= 001\ 110.000\ 111_2$
$= 1\ 110.000\ 111_2$

And we can use a similar technique to convert from binary to octal.

Example: Convert the following binary numbers to octal:

(a) 11110_2

(b) 1010000100_2

Answer: First, group the binary digits into threes, starting from the radix point and moving outwards. Add leading zeros as appropriate. Then rewrite each set of three into the corresponding digit in octal.

(a) $11110_2 = 011\ 110._2 = \overbrace{011}^{3}\ \overbrace{110}^{6}{}_2 = 36_8$

(b) $1010000100_2 = 001\ 010\ 000\ 100._2 = \overbrace{001}^{1}\ \overbrace{010}^{2}\ \overbrace{000}^{0}\ \overbrace{100}^{4}{}_2 = 1204_8$

The reason we want to group from the radix point instead of just saying "group from the right" is that we want to expand this idea to include non-integer conversions.

Example: Convert the following binary numbers to octal:

(a) 11.11_2

(b) 1100.110101_2

Answer: First, group the binary digits into threes, starting from the radix point, adding extra leading and/or trailing zeros as appropriate. Then rewrite each set of three into the corresponding digit in octal.

(a) $11.11_2 = 011.110_2 = \overbrace{011}^{3}\ .\ \overbrace{110}^{6}{}_2 = 3.6_8$

(b) $1100.110101_2 = 001\ 100.110\ 101_2 = \overbrace{001}^{1}\ \overbrace{100}^{4}\ .\ \overbrace{110}^{6}\ \overbrace{101}^{5}{}_2 = 14.65_8$

1.5.2 Converting Between Binary and Hexadecimal

Similarly, let's first count from zero to fifteen in both hexadecimal and binary.

hexadecimal	binary
0_{16}	0_2
1_{16}	1_2
2_{16}	10_2
3_{16}	11_2
4_{16}	100_2
5_{16}	101_2
6_{16}	110_2
7_{16}	111_2

hexadecimal	binary
8_{16}	1000_2
9_{16}	1001_2
A_{16}	1010_2
B_{16}	1011_2
C_{16}	1100_2
D_{16}	1101_2
E_{16}	1110_2
F_{16}	1111_2

If we add leading zeros where necessary to bring all binary numbers to four digits, we can see that

$$\begin{aligned} 9_{16} &= 1001_2 \\ D_{16} &= 1101_2 \\ 2_{16} &= 0010_2 \end{aligned}$$

Then if we look at the number $9D2_{16}$ written in binary, and split the digits into groups of four,

$$\begin{aligned} 9D2_{16} &= \overbrace{1001}^{9}\overbrace{1101}^{D}\overbrace{0010}^{2}{}_2 \\ &= 1001\ 1101\ 0010_2 \end{aligned}$$

Once again, we can remove the spaces from the binary number if we wish, but keeping them makes the number easier to read. Grouping the digits of binary numbers in either groups of two or three is acceptable, always remembering to start from the radix point and work outward.

Let's work through more examples.

> **Example:** Convert the following hexadecimal numbers to binary.

(a) $3A_{16}$

(b) $F02C_{16}$

(c) $2B.B6_{16}$

(d) $4C.70A_{16}$

Answer:

(a) $$\begin{aligned} 3A_{16} &= \overbrace{0011}^{3}\overbrace{1010}^{A}{}_2 \\ &= 11\ 1010_2\ \text{, dropping the leading zeros} \end{aligned}$$

(b) $$\begin{aligned} F02C_{16} &= \overbrace{1111}^{F}\overbrace{0000}^{0}\overbrace{0010}^{2}\overbrace{1100}^{C}{}_2 \\ &= 1111\ 0000\ 0010\ 1100_2 \end{aligned}$$

(c) $$\begin{aligned} 2B.B6_{16} &= \overbrace{0010}^{1}\overbrace{1011}^{B}.\overbrace{1011}^{B}\overbrace{0110}^{6}{}_2 \\ &= 10\ 1011.1011\ 011_2\ \text{, dropping the leading and trailing zeros} \end{aligned}$$

(d) $$\begin{aligned} 4C.70A_{16} &= \overbrace{0100}^{4}\overbrace{1100}^{C}.\overbrace{0111}^{7}\overbrace{0000}^{0}\overbrace{1010}^{A}{}_2 \\ &= 100\ 1100.0111\ 0000\ 101_2 \end{aligned}$$

Example: Convert the following binary numbers to hexadecimal.

(a) 11110_2

(b) 1010001111_2

(c) 101010.010101_2

Answer: First, group the binary digits into fours, starting from the radix point. Then rewrite each set of four into the corresponding digit in hexadecimal.

(a) $11110_2 = 1\ 1110_2 = \overbrace{1}^{1}\overbrace{1110}^{14}{}_2 = 1E_{16}$

(b) $1010001111_2 = 10\ 1000\ 1111_2 = \overbrace{10}^{2}\overbrace{1000}^{8}\overbrace{1111}^{15}{}_2 = 28F_{16}$

(c) $101010.010101_2 = 10\,1010.0101\,01_2 = \overbrace{10}^{2}\overbrace{1010}^{A}.\overbrace{0101}^{5}\overbrace{0100}^{4}{}_2 = 2A.54_{16}$

1.5.3 Converting Between Octal and Hexadecimal

The fastest way to do this is to convert into binary first, then regroup the binary digits.

Example: Convert the following octal numbers to hexadecimal.

(a) 72_8

(b) 333_8

(c) 55.06_8

Answer:

(a) $72_8 = \overbrace{111}^{7}\overbrace{010}^{2}{}_2$, giving you groups of 3

$= 111010_2$

$= 11\ 1010_2$, changing to groups of 4

$= \overbrace{11}^{3}\overbrace{1010}^{A}{}_2$

$= 3A_{16}$

(b) $333_8 = \overbrace{011}^{3}\overbrace{011}^{3}\overbrace{011}^{3}{}_2$

$= 011\ 011\ 011_2$

$= 11011011_2$

$= 1101\ 1011_2$

$= \overbrace{1101}^{D}\overbrace{1011}^{B}{}_2$

$= DB_{16}$

(c) $55.06_8 = \overbrace{101}^{5}\,\overbrace{101}^{5}\,.\,\overbrace{000}^{0}\,\overbrace{110}^{6}\,_2$
$= 101\ 101.000\ 110_2$
$= 101101.000110_2$
$= 10\ 1101.0001\ 10_2$
$= \overbrace{0010}^{2}\overbrace{1101}^{D}.\overbrace{0001}^{1}\overbrace{1000}^{8}{}_2$
$= 2D.18_{16}$

Example: Convert the following hexadecimal numbers to octal.

(a) $9A_{16}$

(b) $4E59_{16}$

(c) $8.EEF_{16}$

Answer:

(a) $9A_{16} = \overbrace{1001}^{9}\overbrace{1010}^{A}{}_2$
$= 10011010_2$
$= 10\ 011\ 010_2$
$= \overbrace{10}^{2}\,\overbrace{011}^{3}\,\overbrace{010}^{2}\,_2$
$= 232_8$

(b) $4E59_{16} = \overbrace{0100}^{4}\overbrace{1110}^{E}\overbrace{0101}^{5}\overbrace{1001}^{9}{}_2$
$= 0100\ 1110\ 0101\ 1001_2$
$= 100111001011001_2$
$= 100\ 111\ 001\ 011\ 001_2$
$= \overbrace{100}^{4}\,\overbrace{111}^{7}\,\overbrace{001}^{1}\,\overbrace{011}^{3}\,\overbrace{001}^{1}\,_2$
$= 47131_8$

(c)
$$
\begin{aligned}
8.EEF_{16} &= \overbrace{1000}^{8}.\overbrace{1110}^{E}\overbrace{1110}^{E}\overbrace{1111}^{F}{}_2 \\
&= 1000.1110\ 1110\ 1111_2 \\
&= 1000.111011101111_2 \\
&= 1\ 000.111\ 011\ 101\ 111_2 \\
&= \overbrace{001}^{1}\ \overbrace{000}^{0}\ .\ \overbrace{111}^{7}\ \overbrace{011}^{3}\ \overbrace{101}^{5}\ \overbrace{111}^{7}{}_2 \\
&= 10.7357_8
\end{aligned}
$$

Exercises for Section 1.5

Convert the following octal numbers to binary:

1. 113_8
2. 20.1_8
3. 1104_8

Convert the following hexadecimal numbers to binary:

4. $2B_{16}$
5. $3C.C_{16}$
6. $29A_{16}$

Convert the following binary numbers to octal:

7. 1100_2
8. 1001100_2
9. 11011.1001_2

Convert the following binary numbers to hexadecimal:

10. 10011_2
11. 1000000_2
12. 1.101111_2

Convert the following octal numbers to hexadecimal:

13. 1.6_8
14. 142_8
15. 24.57_8
16. 5002_8

Convert the following hexadecimal numbers to octal:

17. $C.2_{16}$
18. $1D07_{16}$
19. $A.2E6_{16}$

Perform the following conversions for non-integer numbers:

20. $E.15_{16}$ to binary
21. 4.702_8 to binary
22. 10.011_2 to hexadecimal
23. 110.1_2 to octal
24. $7B.B_{16}$ to octal
25. 4.1702_8 to hexadecimal

Answers to Section 1.5 Exercises

1. $113_8 = 1001011_2$
2. $20.1_8 = 10\ 000.001_2$
3. $1104_8 = 1001000100_2$
4. $2B_{16} = 101011_2$
5. $3C.C_{16} = 11\ 1100.11_2$
6. $29A_{16} = 1010011010_2$
7. $1100_2 = 14_8$
8. $1001100_2 = 114_8$
9. $11011.1001_2 = 33.44_8$
10. $10011_2 = 13_{16}$
11. $1000000_2 = 40_{16}$
12. $1.101111_2 = 1.BC_{16}$
13. $1.6_8 = 1.C_{16}$
14. $142_8 = 62_{16}$
15. $24.57_8 = 14.BC_{16}$
16. $5002_8 = A02_{16}$
17. $C.2_{16} = 14.1_8$
18. $1D07_{16} = 16407_8$
19. $A.2E6_{16} = 12.1346_8$
20. $E.15_{16} = 1110.0001\ 0101_2$
21. $4.702_8 = 100.111\ 000\ 01_2$
22. $10.011_2 = 2.6_{16}$
23. $110.1_2 = 6.4_8$
24. $7B.B_{16} = 173.54_8$
25. $4.1702_8 = 4.3C2_{16}$

Mixed Practice

Convert the following numbers to the indicated base. Give exact answers unless directed otherwise. Show your work.

1. $5B2_{16}$ to binary
2. 0.12_{16} to decimal
3. 5392 to octal
4. 19.5625 to binary
5. 11010.01011_2 to octal
6. 703.1_8 to decimal
7. 0.33 to hexadecimal
8. 33.72_8 to hexadecimal
9. 44.02_5 to decimal
10. 101010.01_2 to hexadecimal
11. 44.02 to base 5
12. 262.8125 to octal

Answers

1. $101\ 1011\ 0010_2$ (the spacing is not necessary, but it makes the result easier to read)
2. 0.0703125
3. 12420_8
4. 10011.1001_2
5. 32.26_8
6. 451.125
7. $0.5\overline{47AE1}_{16}$
8. $1B.E8_{16}$
9. 24.08
10. $2A.4_{16}$
11. $134.00\overline{2}_5$
12. 406.64_8

Chapter 2

Logic

2.1 Introduction to Logic

2.1.1 Propositions

In logic, a proposition is a statement that is either true or false but not both. The statement must also be unambiguous.

Examples of statements that **are** propositions:

(a) Trevor Noah is the host of the Daily Show on the Comedy Network.

(b) Lego Star Wars is a video game.

(c) The number π is exactly equal to 3.

A proposition can clearly be false, as in the last statement, while still being a proposition. Examples of statements that are **not** propositions:

(a) Will you do your homework tonight?

(b) Please pass the butter.

(c) She was late for class this morning.

The first is not a proposition because questions cannot be propositions. (Note that the **answer** to the question may very well be a proposition.) The second one is a command and cannot be said to be either true or false. The third of these examples is not a proposition because, taking the statement on its own, the truth value depends on who "she" is. If, however, that statement were expanded to become, "My roommate's name is Laura and she was late for class this morning," then "she" is clearly defined to be Laura and the whole sentence is a proposition.

Taking this idea one step further, we can consider "she" in the third example to behave like a variable, and whether the full statement "she was late" is true or false must depend on what the value of the variable "she" is. Similarly, in programming it is very common to evaluate the value (true/false) of propositions like "$x = 3$" or "$y < 5$" in statements like:

```
if x = 3 then print ''Hello World''
```

provided that, like she/Laura, the value of x has previously been defined.

Since writing propositions out using English sentences is unwieldy, we frequently use variables to denote propositions. In symbolic logic, we usually use the letters p, q, r, s, t, etc., for propositions. Each of these variables can then have one of two values, true or false. For example, let $p =$ "Lego Star Wars is a video game" and $q =$ "The number π is exactly equal to 3." In this instance, the proposition p is true, since there is a video game called Lego Star Wars and the proposition q is false, since π is the irrational number 3.1415926... which does not repeat and does not terminate.

2.1.2 Operators

"not"

The negation of any proposition p is called "not p" and is written using the tilde symbol, $\sim p$. The tilde can be found on a standard keyboard as the shifted key to the left of the 1. You may also see negations in logic written using this symbol, $\neg p$, or using a bar or overline, $\overline{p}$. In computing, you will also see $!p$ as the negation. We will use the $\sim p$ notation because it uses a character on a conventional keyboard, so is easier to type.

Note that you should be a little careful when negating sentences. For example, the negative of "Pat is happy" is not "Pat is unhappy". There are many other emotions that Pat could have (anger, fear, boredom, etc.). If the first statement is false, then its negation must be true, so between the two you need to cover all possible situations that could arise. It would be safe to say that the negation of "Pat is happy" is that "Pat is not happy", though.

Example: Are these two sentences negatives of each other?

- "The number of students in Math 155 is even."
- "The number of students in Math 155 is odd."

Answer: Yes, these two are negatives of each other. Since we never have fractions of students in class, the number of students must be either zero or a natural number (in other words, a whole number). Since whole numbers and natural numbers are either even or odd, these two sentences cover all bases and are negatives of each other. (Yes, zero is an even number.)

Example: Are these two sentences negatives of each other?

- "Pat's Visa account balance is positive."
- "Pat's Visa account balance is negative."

Answer: No, these two statements are not negatives of each other. There is a third possible case, "Pat's Visa account balance is zero", since zero is an unsigned number. So the two statements above don't cover all options. However, if the second statement read "Pat's Visa account balance is negative or zero", then the second statement would be the negation of the first one.

2.1.3 Combining Two or More Propositions Using Connectives

Propositions may be combined using logical operators called **connectives**, and the result is called a **compound proposition**. There are three basic connectives that we will study: "and", "or", and "exclusive or". (Oddly enough, the "not" operator is also called a connective, even though it acts on only one entity rather than joining two.)

"and"

If we connect the propositions p and q with "and" (also called the **conjunction**), then "p and q" is true if both p and q are true. The symbol for "and" is $\wedge$, so "p and q" is written $p \wedge q$.

Example: Under what conditions is the statement "Pat does her marking and goes to a movie" true?

> Answer: "Pat does her marking and goes to a movie" is true if and only if Pat both does her marking and goes to a movie. If she does one or the other **but not both**, then the statement "Pat does her marking and goes to a movie" is false. It's also false if she does neither action.

"or"/"inclusive or"

If we connect the propositions p and q with "or" (also called the **inclusive disjunction**), then "p or q" is true if either p or q or both are true. The symbol for "or" is $\vee$, so "p or q" is written $p \vee q$.

> **Example:** Under what conditions is the statement "Pat does her marking or goes to a movie" true?
>
> Answer: "Pat does her marking or goes to a movie" is true if and only if at least one of the conditions is true. If she does her marking, then the compound proposition is true whether or not she goes to a movie, and if she goes to a movie, then the statement is true whether or not she does her marking. To really spell it out, "Pat does her marking or goes to a movie" is true if **any** of the following are true:
>
> - Pat does her marking and also goes to a movie
> - Pat does her marking but does not go to a movie
> - Pat does not do her marking but does go to a movie
>
> The only way "Pat does her marking or goes to a movie" is false **only** when Pat does not do her marking **and** does not go to a movie.

"XOR"/"exclusive or"

If we connect the propositions p and q with "exclusive or" (also called the **exclusive disjunction** and frequently written as `XOR`), then "p `XOR` q" is true if either p or q **but not both** are true. The symbol for "exclusive or" is $\oplus$, so "p `XOR` q" is written $p \oplus q$.

"or" vs. "XOR"

In ordinary English, the word "or" can mean either the "inclusive or" or the "exclusive or", and it is usually up to the reader/listener to decide which one was meant from the context.

> **Example:** Which "or" is meant in the following English sentences/phrases?
>
> (a) "Would you like milk or sugar in your tea?"
>
> (b) "Wanted dead or alive"
>
> Answer: For (a), the answer could easily be "milk", "sugar", "both", or "neither". Since "both" is an option, "inclusive or" is clearly meant.
>
> For (b), the person who is wanted in one of these two states will either be dead or alive but not both, so "exclusive or" is the best interpretation.[1]

To unambiguously state which "or" is meant in English, the word "or" can be replaced by slightly wordier constructions. The sentence "Would you like milk or sugar or both in your tea?" makes it clear that the "inclusive or" is meant. Replacing "or" by "and/or" has the same result. Using "either ... or" or the phrase "but not both" are signals that the "exclusive or" is meant.

In general, if a statement is ambiguous, it is best to seek clarification. If that is not possible, then assuming that "or" means the "inclusive or" is generally the safest bet. For the rest of this course, we will use "or" to mean the "inclusive or".

> **Example:** Under what conditions is the statement "Pat does her marking or goes to a movie but not both" true?
>
> Answer: "Pat does her marking or goes to a movie but not both" is true if and only if **only** one of the conditions is true. If she does her marking, then she cannot also go to a movie. If she goes to a movie, then she cannot also do her marking. The exclusive or means that she cannot do both and she cannot do neither.

Logical Propositions and the Order of Operations

When you are doing arithmetic, to evaluate the expression

$$4 + 3 \times 2^2$$

[1] Unless, of course, there is a zombie apocalypse.

you need to know which operation (addition, multiplication, exponentiation) should come first. In the same way, there is an order of operations in logic:

- negation, $\sim$, is done first
- "and", $\wedge$, is done next
- "or", $\vee$, is done last

and brackets () override the default order. If you like, you can think of "not" like exponents, "and" like multiplication, and "or" like addition (more on this later).

To evaluate the proposition $p \vee q \wedge r$, you would note that "and", $\wedge$, comes before "or", so you'd evaluate $q \wedge r$ first, and then "or" it with p. So $p \vee q \wedge r$ is the same as $p \vee (q \wedge r)$.

To evaluate the proposition $\sim q \wedge p$, you'd negate the q first, and then "and" with p. So $\sim q \wedge p$ is the same as $(\sim q) \wedge p$. If you want the "and" to come first, then override with brackets: $\sim(q \wedge p)$. We note that $\sim q \wedge p$ is not the same as $\sim(q \wedge p)$.

We will be practicing this skill once we get to truth tables.

Exercises for Section 2.1

State whether the following sentences are propositions.

1. On September 6, 2006, mathematicians proved that $2^{32582657} - 1$ was a prime number.
2. Will you marry me?
3. Python is her favourite computing language.
4. What is your favourite computing language?
5. Please bring me a textbook.
6. The University of Victoria is located in Alberta.

Let p be "Rich is seven feet tall" and q be "Susan has brown hair." Translate the following English sentences into logical notation.

7. Rich is seven feet tall or he is seven feet tall.
8. Either Rich is not seven feet tall or Susan does not have brown hair.
9. It is not true that Rich is seven feet tall or Susan has brown hair.
10. Rich is seven feet tall and Susan has brown hair.
11. Either Rich is seven feet tall or Susan does not have brown hair, but not both.

Which type of "or", inclusive or exclusive, is meant in the following English sentences?

12. Do you want to sit inside or outside?
13. Have you seen the latest Harry Potter or Transformers movie?
14. I think I'll get an A or a B in the course.
15. Is that the correct answer or not?
16. We need someone who speaks French or German.

Let p be "The moon is made of green cheese" and q be "The earth is made of green cheese." Translate the following English sentences into logical notation.

17. Either the moon is made of green cheese or both the moon and the earth are made of green cheese.
18. The earth is made of green cheese and either the earth or the moon is made of green cheese.
19. Either the earth is made of green cheese while the moon is not, or the moon is made of green cheese.
20. The earth is made of green cheese and either the moon is made of green cheese or the earth is not.

Let $p =$ "Jane did her homework" and $q =$ "Jane went for a jog." Translate the following logical propositions into English sentences.

21. $p \wedge q$
22. $\sim(p \wedge q)$
23. $q \wedge \sim p$
24. $\sim q \vee \sim p$
25. $\sim(\sim p)$ (that's "not(not p)")
26. $q \oplus \sim q$

For each pair of sentences below, is the second sentence the negation of the first?

27. Pat owes Peter money. Peter owes Pat money.
28. The number of students in Math 155 is greater than 25. The number of students in Math 155 is less than 25.
29. Pat, the math instructor, is rich. Pat, the math instructor, is poor.

Answer the questions given the following situations. If you cannot answer the question, state whether "the situation is not possible" or "there's not enough information."

30. Jane went for a jog and did her homework. Did she go for a jog?
31. Jane went for a jog or did her homework. Did she not do her homework?

32. Jane went for a jog. Did she go for a jog and do her homework?
33. Jane did not go for a jog. Did she go for a jog and do her homework?

Answers to Section 2.1 Exercises

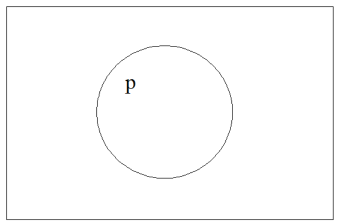
1. Yes
2. No
3. No
4. No
5. No
6. Yes
7. $p \vee p$
8. From the context, you could go with either $\sim p \vee \sim q$ or $\sim p \oplus \sim q$.
9. $\sim(p \vee q)$
10. $p \wedge q$
11. $p \oplus \sim q$
12. exclusive (you usually don't sit both inside and outside at the same time)
13. inclusive (you could have seen both)
14. exclusive (you can only get one mark for the course, so it's one or the other but can't be both)
15. exclusive (it can't both be the correct answer and not the correct answer at the same time)
16. inclusive (it's possible that someone speaks both languages)
17. $p \vee (p \wedge q)$
18. $q \wedge (q \vee p)$
19. $(q \wedge \sim p) \vee p$
20. $q \wedge (p \vee \sim q)$
21. Jane did her homework and went for a jog.
22. It is not true that Jane both did her homework and went for a jog.
23. Jane went for a jog and Jane did not do her homework.

24. Jane did not go for a jog or she didn't do her homework.
25. It is not true that Jane didn't do her homework.
26. Either Jane went for a jog or she didn't, but not both.
27. No. (They could just be even, not owing each other anything.)
28. No. (What if there were exactly 25 students in the class?)
29. No. (Maybe Pat is middle class, so is neither rich nor poor?)
30. Yes.
31. Not enough info. Depends on whether she went for a jog. If she did go for a jog, she could have not done her homework. But if she didn't go for a jog, she must have done her homework for sure.
32. Not enough info. Depends on whether she did her homework.
33. No.

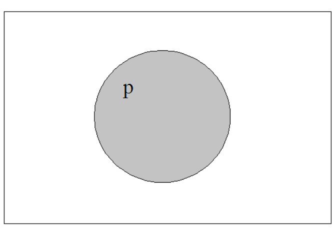
2.2 Venn Diagrams

2.2.1 Venn Diagrams with One Proposition

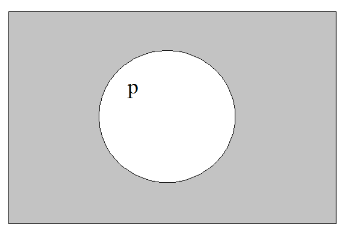
One way to visual operations on propositions is to use a Venn diagram. Although Venn diagrams are more commonly used with sets, there are many commonalities between the operations on sets and on logical propositions. The Venn diagram for a single logical proposition p is shown below.



In this diagram, the rectangle stands for the universe, while the circle denotes the logical proposition p. We then shade in regions of the diagram to indicate the regions of interest. For example, when we want to indicate the proposition p, we shade the inside of the circle, as shown in the left diagram. If instead we want to show the proposition $\sim p$, we shade outside of the circle, as in the diagram to the right.



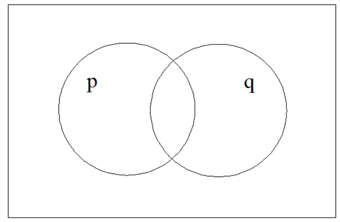
shading for p



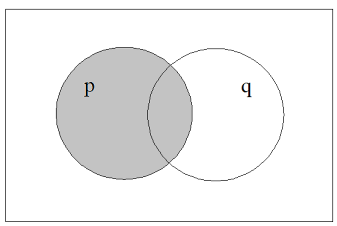
shading for $\sim p$

2.2.2 Venn Diagrams with Two Propositions

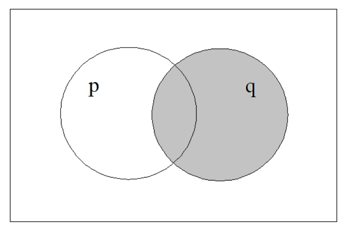
Venn diagrams with only one proposition don't generally contain much information, as it's usually pretty easy to visualize what p and $\sim p$ mean when you only have the one proposition. Where it gets more interesting is when you have propositions p and q in the same diagram, as you can see in the next diagram.



Let's try doing some shading to represent operations on the propositions p and q. To begin with, let's examine the shading for p, as shown in the left diagram below. It looks very similar to the shading for the one-proposition diagram, but you should notice that in order to shade in all of p, **two** regions have been shaded in: the crescent-moon shaped part which represents the part of p that **does not** overlap with q and the lozenge-shaped part which represents the part of p that **does** overlap with q. Similarly, q is shown in the diagram below on the right.



shading for p

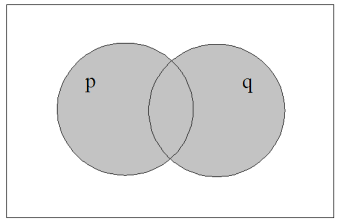


shading for q

Now, if you were to join p and q by an operator such as "and" or "or", then the way to do it is to consider all four regions of the diagram:

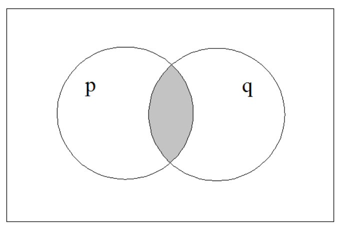
- the left crescent-moon shape belonging to p but not q
- the right crescent-moon shape belonging to q but not p
- the lozenge shape which is the overlap of both p and q
- the remaining region which is neither in p nor q

For example, if you wished to show the Venn diagram for $p \vee q$, recall that "or" means "one or the other or both". You would consider the diagrams for p and q above, and any regions that are shaded in **either** diagram or **both** diagrams will be shaded in for $p \vee q$. So, the parts that would be shaded in the resulting diagram would be the left and right crescent-moon shapes plus the lozenge.



$p \vee q$

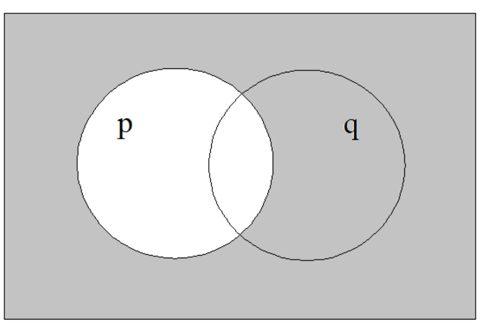
To show $p \wedge q$, you need to shade those regions that are shaded in **both** of the p and q diagrams. This means that you'd shade in the lozenge shaped region, as you can see below.

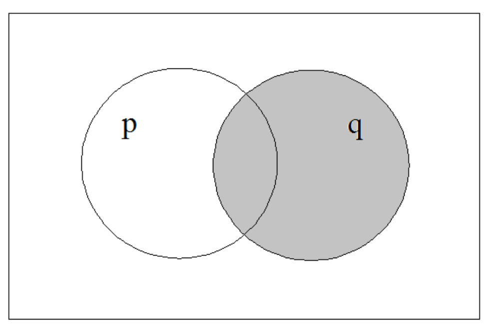


$p \wedge q$

2.2.3 More complications

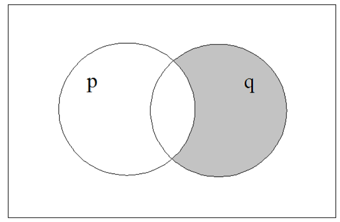
Suppose you wished to shade in a Venn diagram to show $\sim p \wedge q$. A straightforward approach is to do it by steps:





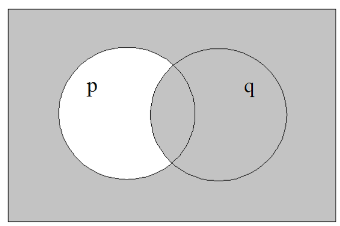
$\sim p$ q

And then to find what happens when we "and" the two diagrams we shade all the regions that are shaded in **both** of the above diagrams.



$\sim p \wedge q$

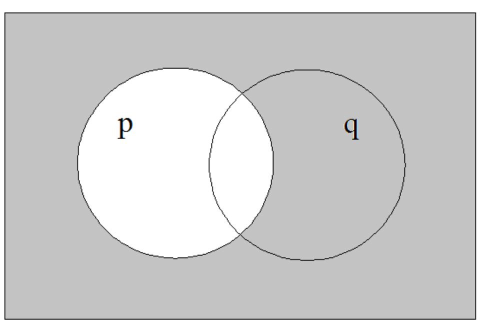
To find $\sim p \vee q$, you'd take the diagrams for $\sim p$ and q, and then shade in the regions that are shaded in for **either** of the above diagrams.



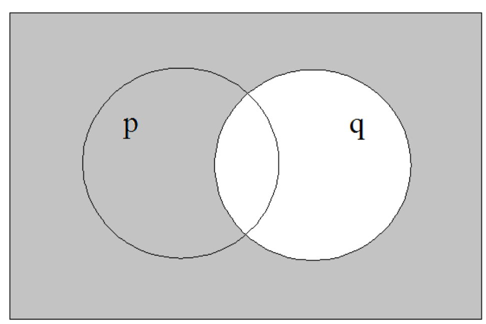
$\sim p \vee q$

Example: Shade in the Venn diagram corresponding to $\sim p \vee \sim q$.

Answer: Here's $\sim p$ and $\sim q$ below.

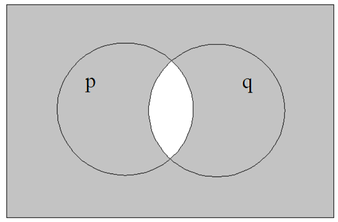


$\sim p$



$\sim q$

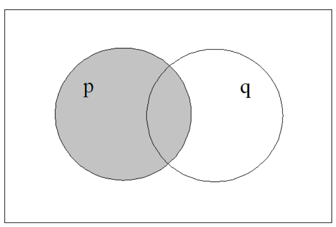
We need to shade in regions that are shaded in for **either** of the above diagrams, to get the following.



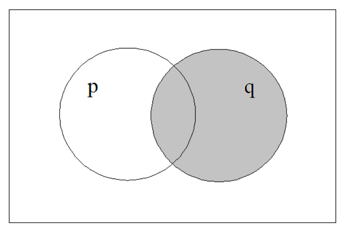
$\sim p \vee \sim q$

2.2.4 Negation and De Morgan's Laws

Consider the proposition $\sim(p \wedge q)$. The brackets mean that we should find $p \wedge q$ first, and then negate it. Let's start by shading the diagrams for p and q:

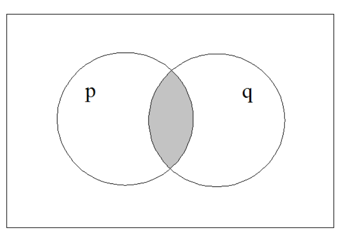


shading for p



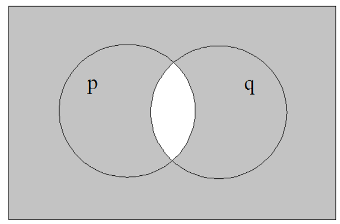
shading for q

To show $p \wedge q$, you need to shade those regions that are shaded in **both** of the p and q diagrams. This means that you'd shade in the lozenge shaped region, as you can see below.



$p \wedge q$

Now to get $\sim(p \wedge q)$ from $p \wedge q$, we take the $p \wedge q$ diagram and negate it. Essentially, we "reverse" the diagram by shading in all previously unshaded regions, and not shading in any previous shaded regions, resulting in the following.

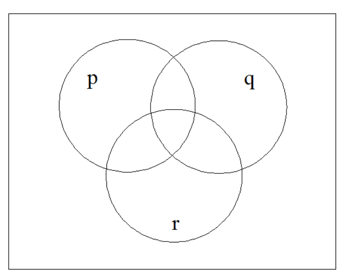


$\sim(p \wedge q)$

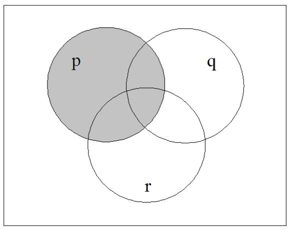
You can see that we get exactly the same result as when we found $\sim p \vee \sim q$. This result, that $\sim p \vee \sim q$ is equivalent to $\sim(p \wedge q)$, is true for all propositions p and q. You could, if you wish, show also that $\sim p \wedge \sim q$ is equivalent to $\sim(p \vee q)$ for all p and q. These two statements are called De Morgan's theorems and we will be revisiting them later.

2.2.5 Venn Diagrams with Three Propositions

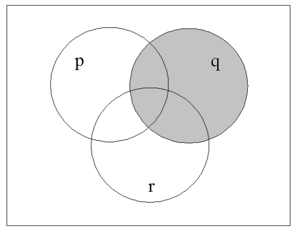
Similarly, we can do Venn diagrams with three propositions, as shown in the next diagram.



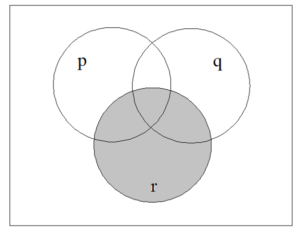
Notice that there is a circle for each set, and that there are regions where some or all of the sets overlap. To find out how to shade the diagram for combinations of sets such as $(p \vee q) \wedge r$, do the shading process in steps. Here's p, q, and r below.



p

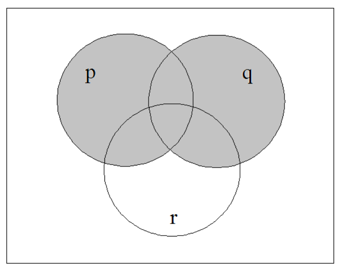


q



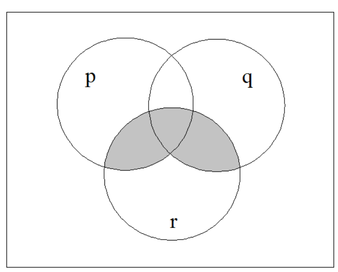
r

Then $p \vee q$ gives



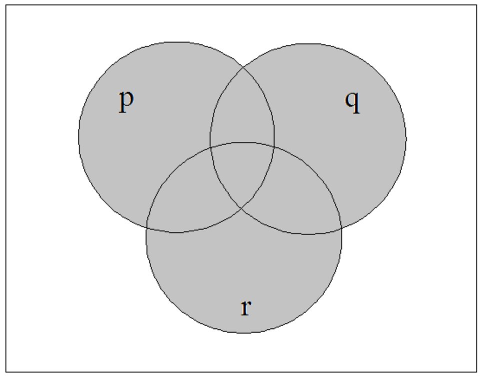
$p \vee q$

and intersecting this with proposition r from above gives

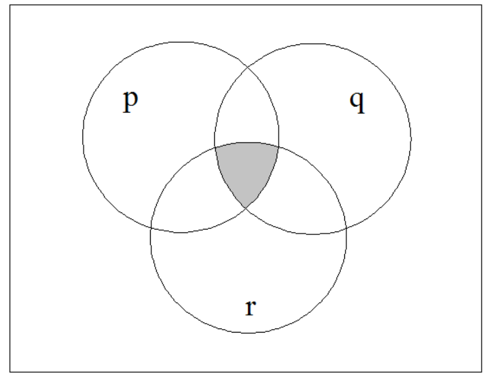


$(p \vee q) \wedge r$

The diagrams for $p \vee q \vee r$ and $p \wedge q \wedge r$ are then given below. (Because the operations are all the same in each expression, I don't need brackets to show the order of operations for these particular cases.)



$p \vee q \vee r$



$p \wedge q \wedge r$

Exercises for Section 2.2

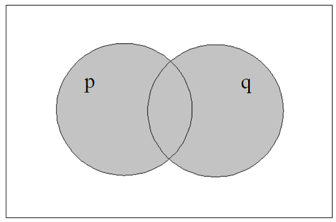
Draw Venn diagrams using two propositions p and q, shading in the appropriate regions for the following situations.

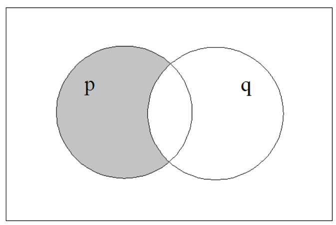
1. $p \vee q$
2. $p \wedge \sim q$
3. $\sim p \wedge \sim q$
4. $\sim(p \wedge \sim q)$ (this would just be the negation of #2)
5. $\sim(p \vee q)$
6. $p \wedge (\sim p \vee q)$
7. $p \vee (p \wedge q)$

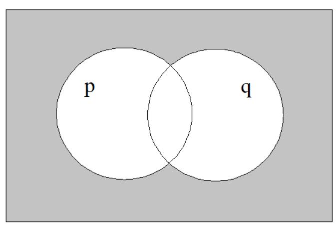
Draw Venn diagrams using three propositions: p, q, and r. Shade in the appropriate regions for the following situations.

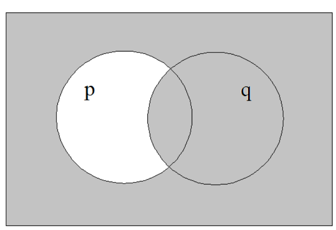
8. $p \vee q \vee r$
9. $(p \wedge q) \vee r)$
10. $p \wedge (q \vee r)$
11. $p \vee \sim q \vee r$
12. $\sim p \wedge q \wedge \sim r$
13. $(p \wedge q) \vee \sim r$
14. $\sim q \wedge (\sim p \vee r)$

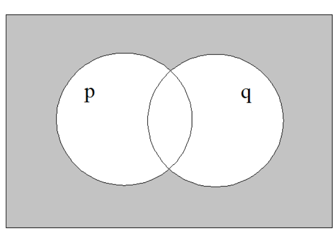
Answers to Section 2.2 Exercises

1. 

2. 

3. 

4. 

5. 

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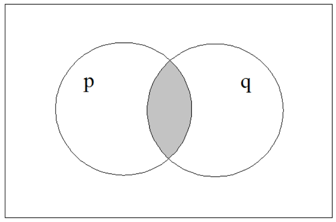

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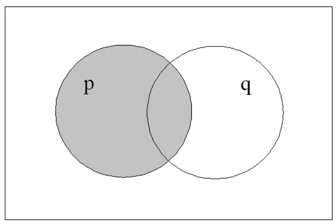

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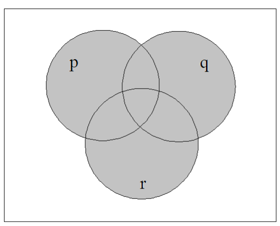

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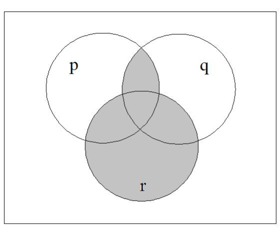

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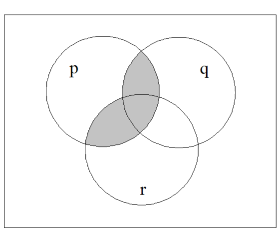
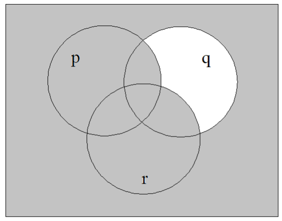
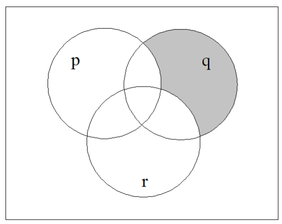
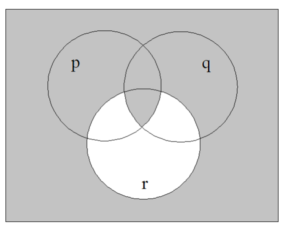
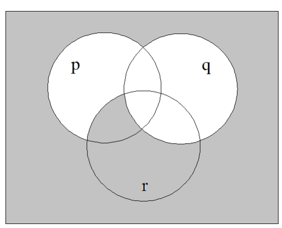


11. 

12. 

13. 

14. 

2.3 Logical Equivalence

2.3.1 Truth Tables

Truth Tables with Two Variables

Let us consider the propositions p and q. Since they are propositions, p is either true or false and q is also either true or false. This leads us to four possible combinations of p and q:

1. p and q are both false
2. p is false and q is true
3. p is true and q is false
4. p and q are both true

We can combine these possibilities into a table called a truth table. We can add further columns to find out what the value of other compound propositions for each combination of p and q as well. Suppose we wished to find out what the truth table was for $p \wedge q$. Then the table would look like the following.

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

For example, when p is false and q is true (the second row, where $p = \text{F}$ and $q = \text{T}$), then $p \wedge q$ is false because one of them is false (they are not both true).

Similarly, the truth table for $p \vee q$ is

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

And if we like, we can combine the two tables above into a single table like so:

p	q	$p \wedge q$	$p \vee q$
F	F	F	F
F	T	F	T
T	F	F	T
T	T	T	T

However, we can also abbreviate the table, changing all Fs to 0s and Ts to 1s. We do this so that there is a good correspondence between these truth tables and the tables we will be learning for sets and Boolean algebra. So another equally correct truth table would be:

p	q	$p \wedge q$	$p \vee q$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

Truth Tables with One Variable

What if we were interested in the truth table for $p \vee p$? Since this truth table contains only one variable p rather than two, it will only have two rows since p can either be true or false (and yes, you could omit the second column if you wish).

p	p	$p \vee p$
0	0	0
1	1	1

Let's now look at how to write the truth table for $p \wedge 1$, where 1 means a statement that is always true. (Be careful! The number 1 is a constant, not a variable! It never takes the value of zero.) The table would look like:

p	1	$p \wedge 1$
0	1	0
1	1	1

We notice that the last column looks like the first, so $p \wedge 1$ has the same values as p. We say, then, that $p \wedge 1$ is **logically equivalent** to p. We'll talk more about logical equivalence in a bit.

Negations in Truth Tables

To negate a variable, we simply switch the value of each entry in that column from its previous value. So if we were interested in the truth table for $p \vee \sim p$, we'd have to negate p. To do this, we'll take every entry in the p column and switch all the zeros to ones and the ones to zeros. We'll then "or" the first and second columns as before.

p	$\sim p$	$p \vee \sim p$
0	1	1
1	0	1

Notice, then, that $p \vee \sim p$ is always true. I hope that makes a certain amount of sense: the proposition "Pat's hair is green or Pat's hair is not green" is a statement that is always true independent of Pat's hair colour.

To negate an expression, we use the same idea and switch the value of each entry in the column for that expression from its previous value. For example, here is the truth table for $\sim(p \wedge 1)$:

p	1	$p \wedge 1$	$\sim(p \wedge 1)$
0	1	0	1
1	1	1	0

You can see that to get the fourth column (which is the negation of the third column), we've just switched the values of the expression in the third column.

Truth Tables with Three Variables

What would the truth table for three propositions look like? We must have eight rows to display all possibilities for p, q, and r. The truth table for $\sim p \wedge (q \vee \sim r)$ would then be

p	q	r	$\sim r$	$q \vee \sim r$	$\sim p$	$\sim p \wedge (q \vee \sim r)$
0	0	0	1	1	1	1
0	0	1	0	0	1	0
0	1	0	1	1	1	1
0	1	1	0	1	1	1
1	0	0	1	1	0	0
1	0	1	0	0	0	0
1	1	0	1	1	0	0
1	1	1	0	1	0	0

It's important to note that the actual order of the rows doesn't matter for the truth table to be complete. However, if you write out the table with the rows in a random order, it's very easy to duplicate one of the previous rows. The duplicate row in and of itself isn't a mistake, but if you stop your table at the correct **total** number of rows, the duplicate means that one of the combinations of your variables is missing, which **is** an error.

Another common mistake is to take a shortcut and start the truth table with one of the columns being, for example, $\sim p$. This is not correct, since truth tables must always start with unnegated variables.

2.3.2 Logical Equivalence

Two logical expressions are said to be **logically equivalent** if they have the same values in their columns in the truth table. We saw in our examples above that $p \vee \sim p$ was logically equivalent to 1 and $p \wedge 1$ was logically equivalent to p. The symbol for "logically equivalent to" is $\Leftrightarrow$, so $p \vee \sim p \Leftrightarrow 1$ and $p \wedge 1 \Leftrightarrow p$.

Example: Is $p \wedge (q \vee r)$ logically equivalent to $(p \wedge q) \vee r$?

Answer:

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$(p \wedge q) \vee r$
0	0	0	0	0	0	0
0	0	1	1	0	0	1
0	1	0	1	0	0	0
0	1	1	1	0	0	1
1	0	0	0	0	0	0
1	0	1	1	1	0	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

No, these two expressions are not logically equivalent because their columns in the truth table, columns 5 and 7, are not identical. This example shows once more that order of operations is important!

Example: Simplify $(p \wedge q) \vee ({\sim}p \wedge q)$.

Answer:

p	q	${\sim}p$	$p \wedge q$	${\sim}p \wedge q$	$(p \wedge q) \vee ({\sim}p \wedge q)$
0	0	1	0	0	0
0	1	1	0	1	1
1	0	0	0	0	0
1	1	0	1	0	1

Notice that the last column is identical to the column for q. Therefore, $(p \wedge q) \vee ({\sim}p \wedge q)$ is logically equivalent to q, which is the simplified logical expression.

Example: Is $p \oplus q$ logically equivalent to $(p \wedge {\sim}q) \vee ({\sim}p \wedge q)$?

Answer:

p	q	$p \oplus q$	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \wedge q$	$(p \wedge \sim q) \vee (\sim p \wedge q)$
0	0	0	1	1	0	0	0
0	1	1	1	0	0	1	1
1	0	1	0	1	1	0	1
1	1	0	0	0	0	0	0

Yes, the two expressions are logically equivalent.

Exercises for Section 2.3

Give the truth tables for the following logical expressions.

1. $p \wedge \sim p$
2. $p \vee 1$
3. $p \wedge \sim q$
4. $\sim(p \vee q)$
5. $p \oplus \sim q$
6. $p \vee (\sim p \wedge q)$
7. $(p \vee q) \wedge r$
8. $p \vee q \vee \sim r$
9. $(p \wedge q) \vee \sim(p \vee \sim q)$
10. $(\sim p \vee \sim q) \wedge (\sim p \vee q)$

Are the two expressions logically equivalent?

11. $\sim(p \wedge q)$ and $\sim p \wedge \sim q$
12. $\sim(p \vee q)$ and $\sim p \wedge \sim q$
13. $p \oplus q$ and $\sim p \oplus \sim q$
14. $p \vee (q \wedge r)$ and $(p \vee q) \wedge r$
15. $p \vee (p \wedge q)$ and p
16. $(p \vee q) \vee r$ and $p \vee (q \vee r)$
17. $p \oplus q$ and $(p \wedge q) \vee (\sim p \wedge \sim q)$

Simplify.

18. $p \wedge p$
19. $p \vee \sim p$
20. $p \wedge 0$
21. $\sim p \oplus p$
22. $(p \oplus q) \wedge (p \oplus \sim q)$

23. $p \vee (p \wedge q)$
24. $q \wedge (p \vee q)$
25. (tricksy) $p \wedge (\sim p \vee q)$
26. (tricksy) $p \vee (\sim p \wedge q)$

Answers to Section 2.3 Exercises

1.

p	$\sim p$	$p \wedge \sim p$
0	1	0
1	0	0

2.

p	1	$p \vee 1$
0	1	1
1	1	1

3.

p	q	$\sim q$	$p \wedge \sim q$
0	0	1	0
0	1	0	0
1	0	1	1
1	1	0	0

4.

p	q	$p \vee q$	$\sim(p \vee q)$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

5.

p	q	$\sim q$	$p \oplus \sim q$
0	0	1	1
0	1	0	0
1	0	1	0
1	1	0	1

6.

p	q	$\sim p$	$\sim p \wedge q$	$p \vee (\sim p \wedge q)$
0	0	1	0	0
0	1	1	1	1
1	0	0	0	1
1	1	0	0	1

7.

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
0	0	0	0	0
0	0	1	0	0
0	1	0	1	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	1
1	1	0	1	0
1	1	1	1	1

8.

p	q	r	$\sim r$	$p \vee q \vee \sim r$
0	0	0	1	1
0	0	1	0	0
0	1	0	1	1
0	1	1	0	1
1	0	0	1	1
1	0	1	0	1
1	1	0	1	1
1	1	1	0	1

9.

p	q	$\sim q$	$p \wedge q$	$p \vee \sim q$	$\sim(p \vee \sim q)$	$(p \wedge q) \vee \sim(p \vee \sim q)$
0	0	1	0	1	0	0
0	1	0	0	0	1	1
1	0	1	0	1	0	0
1	1	0	1	1	0	1

10.

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim p \vee q$	$(\sim p \vee \sim q) \wedge (\sim p \vee q)$
0	0	1	1	1	1	1
0	1	1	0	1	1	1
1	0	0	1	1	0	0
1	1	0	0	0	1	0

11.

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
0	0	0	1	1	1	1
0	1	0	1	1	0	0
1	0	0	1	0	1	0
1	1	1	0	0	0	0

No, because the 4th and 7th columns are not the same.

12.

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

Yes, because the 4th and 7th columns are identical.

13.

p	q	$p \oplus q$	$\sim p$	$\sim q$	$\sim p \oplus \sim q$
0	0	0	1	1	0
0	1	1	1	0	1
1	0	1	0	1	1
1	1	0	0	0	0

Yes, because the 3rd and 6th columns are identical.

14.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$(p \vee q) \wedge r$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	1	0
0	1	1	1	1	1	1
1	0	0	0	1	1	0
1	0	1	0	1	1	1
1	1	0	0	1	1	0
1	1	1	1	1	1	1

No, because the 5th and last columns are not identical.

15.

p	q	$p \wedge q$	$p \vee (p \wedge q)$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

Yes, because the first and last columns are identical.

16.

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	1	1	1	1
0	1	1	1	1	1	1
1	0	0	1	1	0	1
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

Yes, because the 5th and last columns are identical.

17.

p	q	$p \oplus q$	$\sim p$	$\sim q$	$p \wedge q$	$\sim p \wedge \sim q$	$(p \wedge q) \vee (\sim p \wedge \sim q)$
0	0	0	1	1	0	1	1
0	1	1	1	0	0	0	0
1	0	1	0	1	0	0	0
1	1	0	0	0	1	0	1

No, because the 3rd and last columns are not identical. (But I think you can see that the last expression is the negation of column 3.)

18.

p	p	$p \wedge p$
0	0	0
1	1	1

This expression is logically equivalent to p. (You can omit the second column for p if you wish.)

19.

p	$\sim p$	$p \vee \sim p$
0	1	1
1	0	1

This expression is logically equivalent to 1.

20.

p	0	$p \wedge 0$
0	0	0
1	0	0

This expression is logically equivalent to 0.

21.

p	$\sim p$	$\sim p \oplus p$
0	1	1
1	0	1

This expression simplifies to 1.

22.

p	q	$p \oplus q$	$\sim q$	$p \oplus \sim q$	$(p \oplus q) \wedge (p \oplus \sim q)$
0	0	0	1	1	0
0	1	1	0	0	0
1	0	1	1	0	0
1	1	0	0	1	0

This expression simplifies to 0.

23.

p	q	$p \wedge q$	$p \vee (p \wedge q)$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

This expression is logically equivalent to p.

24.

p	q	$p \vee q$	$q \wedge (p \vee q)$
0	0	0	0
0	1	1	1
1	0	1	0
1	1	1	1

This expression simplifies to q.

25.

p	q	$\sim p$	$\sim p \vee q$	$p \wedge (\sim p \vee q)$
0	0	1	1	0
0	1	1	1	0
1	0	0	0	0
1	1	0	1	1

This expression is logically equivalent to $p \wedge q$.

26.

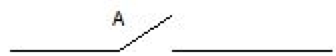
p	q	$\sim p$	$\sim p \wedge q$	$p \vee (\sim p \wedge q)$
0	0	1	0	0
0	1	1	1	1
1	0	0	0	1
1	1	0	0	1

This expression simplifies to $p \vee q$.

2.4 Boolean Algebra

2.4.1 Logic Circuits

A logic circuit or digital circuit is an electrical circuit based on a discrete number of voltage levels, usually two. Two-level circuits usually have one voltage set at zero volts, and the circuit then behaves like a switch, being either **on** or **off**. A nice diagram for a switch looks like this:



so that when the switch is open, as if the diagram, no current flows and the switch is **off**. When the switch closes and there's a clear path from the left side to the right side, the switch is **on**.

A digital circuit then makes logical decisions, based on the input to the circuit. The simplest logic circuits are called **gates**. Physically, a gate is a transistor circuit which takes one or more voltage inputs and gives a single voltage output.

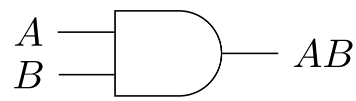
One way to represent the action of a gate is by using a truth table. As usual in a truth table, all possible combinations of the input voltages are given, as well as the output of the gate for each set of inputs. Each input voltage is given a symbol, such as A. When the input signal is off, the value of A is given as 0, and when it's on, the value of A is 1. This then looks exactly like the truth tables we studied with logical propositions.

"and" gate

The **switch representation** of an "and" gate looks like this:

It is a series circuit, and both switches must be closed (on) for the circuit to be complete. You can see, then, that this is the same as "A and B", since "A and B" is true when both A is true and B is true.

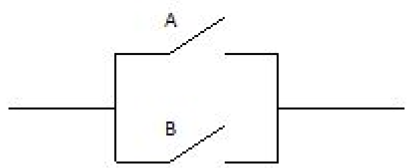
Another common representation is the **gate representation**, which looks like this:



In symbols, we write "A and B" as $A \cdot B$ or just AB.

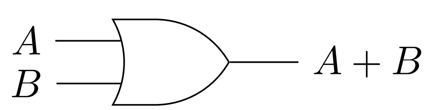
"or" gate

The **switch representation** of an "or" gate looks like this:



It is a parallel circuit, and at least one switch must be closed (on) for the circuit to be complete from left to right. You can see, then, that this is the same as "A or B", since "A or B" is true when either A is true or B is true or both.

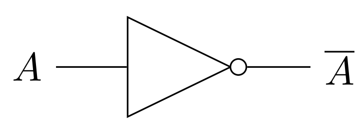
Another common representation is the **gate representation**, which looks like this:



In symbols, we write "A or B" as $A + B$.

"not" gate

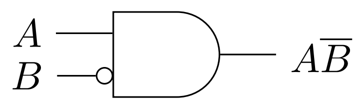
The "not" gate, or inverter, has the diagram



and we write (not-A) as $\overline{A}$. If the negation happens in combination with another gate, we usually omit the triangle and just have a little circle to show the negation, as in the next example.

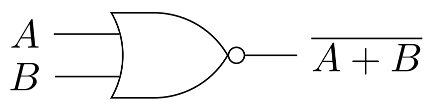
2.4.2 Gate Representations of Logic Circuits

The gate representation of the logic circuit for $A\overline{B}$ is then

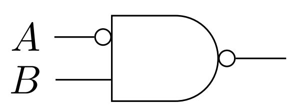


with the round circle on the input B negating it, so that the two inputs to the "and" gate (the semicircle) are then A and $\overline{B}$.

The gate representation for $\overline{A+B}$ is then



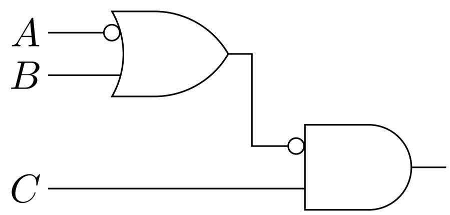
with the "or" gate giving $A+B$, which the little round circle then negates.[2]

> **Example:** What is the logic circuit expression for the following gate diagram?
>
> 
>
>
> Answer: $\overline{\overline{A}B}$

We can also have multiple "and" or "or" gates, or combinations of them, in our diagrams, as in the following example.

[2]Strictly speaking, the "or" gate together with a negation on the output is called a "nor" gate, just like the word "nor" in English. There is also a "nand" gate, which does not have an analogous word in the English language. But as a gentle introduction to gate diagrams, we will only be using the three basic gates: "and", "or", and "not".

Example: What is the logic circuit expression for the following gate diagram?



Answer: $\overline{\overline{A} + B}\ C$

2.4.3 Boolean Algebra

The symbols used for circuits, AB, $A + B$, and $\overline{A}$, are the same symbols as used in Boolean algebra. In this type of algebra, each variable (A, B, etc.) can only have two values, 0 and 1.

Truth tables in Boolean algebra then look very similar to the truth tables that we've studied in logic. For example, the truth table showing AB and $A + B$ is:

A	B	AB	$A+B$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

If you have more than one operation happening in a Boolean expression, the order of operations is very similar to the order of operations in arithmetic. For example, if you have the logical expression $AB + C$, in arithmetic you multiply before you add. In Boolean algebra, you "and" before you "or", just as in symbolic logic. But here the negation sign behaves in the same way as brackets do.

For example, here is the order in which you would evaluate the following Boolean expressions.

$AB + C$	First evaluate AB, then do the "$+C$".
$A + BC$	First evaluate BC, then "or" with A.
$\overline{A}B$	First evaluate $\overline{A}$, then "and" with B.
$\overline{A + B}$	First evaluate $A + B$, then negate the result.
$A(B + C)$	First evaluate the expression in the brackets, then "and" with A.
$A\ \overline{B + C}$	First evaluate the expression $B + C$, then negate it, then "and" with A.

Truth tables can then be used to demonstrate logical equivalence between Boolean expressions.

Example: Is $AB + C$ logically equivalent to $A(B + C)$?

Answer:

A	B	C	AB	$AB + C$	$B + C$	$A(B + C)$
0	0	0	0	0	0	0
0	0	1	0	1	1	0
0	1	0	0	0	1	0
0	1	1	0	1	1	0
1	0	0	0	0	0	0
1	0	1	0	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

And as the 5^{th} and 7^{th} columns aren't identical, these two expressions aren't logically equivalent. Once again, order of operations matters.

2.4.4 Boolean Syntax in Python

Python allows you to perform logical operations on Boolean variables in the way that you would expect, as you can see in the accompanying figure.

```
>>> True or False
True
>>> True and False
False
>>> not True
False
>>> not False
True
>>> 1 and 0
0
>>> 1 or 0
1
>>> 
```

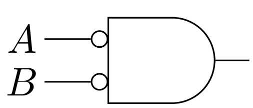
Exercises for Section 2.4

Draw the gate representation for the following logical expressions.

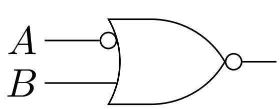
1. $A + \overline{B}$
2. $\overline{A + B}$
3. $\overline{A}B$
4. $\overline{A}\ \overline{B}$
5. $\overline{A + \overline{\overline{B}}}$
6. $A\overline{B} + C$
7. $A(B + \overline{C})$
8. $\overline{AB}C$
9. $\overline{\overline{\overline{A}}\ \overline{\overline{B}} + \overline{C}}$

Write the Boolean expression which corresponds to the following gates.

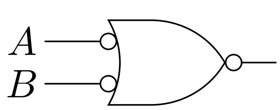
10.



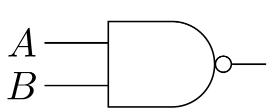
11.



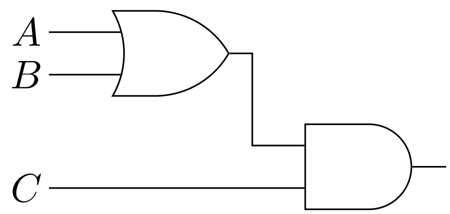
12.



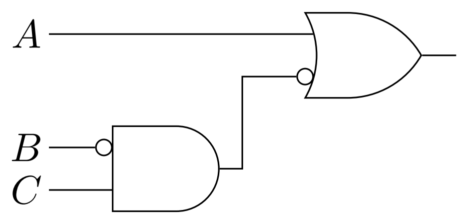
13.



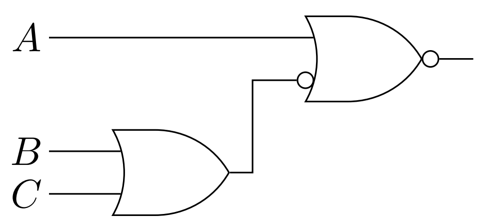
14.



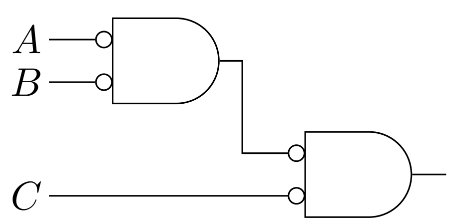
15.



16.



17.



Give the truth tables for the following expressions.

18. $A\overline{A}$

19. $A + 1$

20. $A\overline{B}$

21. $\overline{A+B}$
22. $A+\overline{A}B$
23. $(A+B)C$
24. $A+B+\overline{C}$

Are the two expressions logically equivalent? Justify your answer by giving a truth table.

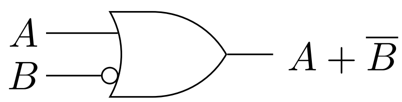
25. $\overline{AB}$ and $\overline{A}\ \overline{B}$
26. $\overline{A+B}$ and $\overline{A}\ \overline{B}$
27. $A+BC$ and $(A+B)C$
28. $A+AB$ and A
29. $(A+B)+C$ and $A+(B+C)$

Simplify the following logical expressions using truth tables.

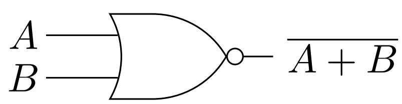
30. AA
31. $A+A$
32. $A+0$
33. $A+AB$
34. $A(\overline{A}+B)$ – this one's a bit trickier! If you're stuck, try writing the truth tables for combinations of A and B, like $(A+B)$ for example, to find one that fits.

Answers to Section 2.4 Exercises

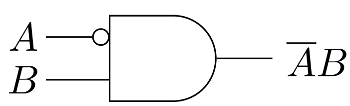
1.



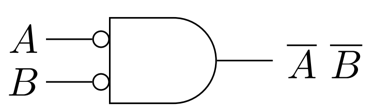
2.



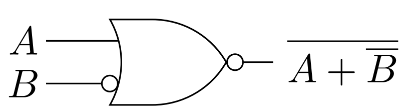
3.



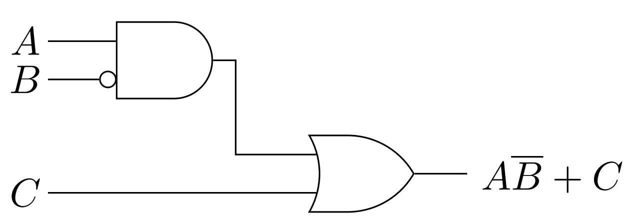
4.



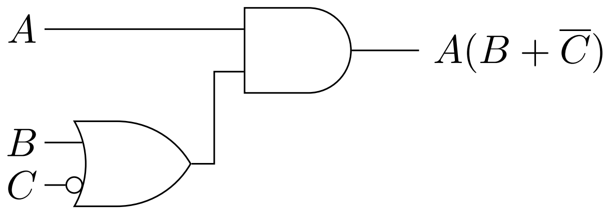
5.



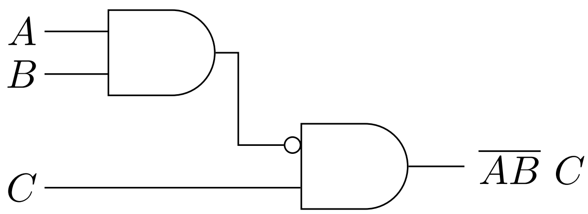
6.



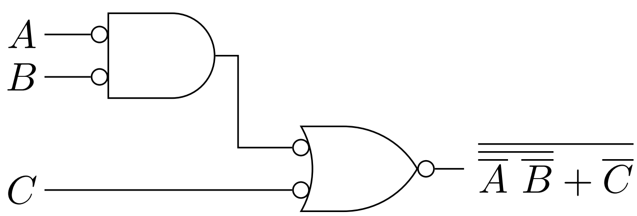
7.



8.



9.



10. $\overline{A}\ \overline{B}$

11. $\overline{\overline{A}+B}$

12. $\overline{\overline{A}+\overline{B}}$

13. $\overline{AB}$

14. $(A+B)\cdot C$

15. $A+\overline{\overline{B}C}$

16. $\overline{A+\overline{B+C}}$

17. $\overline{\overline{A}\ \overline{B}}\ \overline{C}$

18.

A	$\overline{A}$	$A\,\overline{A}$
0	1	0
1	0	0

19.

A	1	$A+1$
0	1	1
1	1	1

20.

A	B	$\overline{B}$	$A\,\overline{B}$
0	0	1	0
0	1	0	0
1	0	1	1
1	1	0	0

21.

A	B	$A+B$	$\overline{A+B}$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

22.

A	B	$\overline{A}$	$\overline{A}\,B$	$A+\overline{A}\,B$
0	0	1	0	0
0	1	1	1	1
1	0	0	0	1
1	1	0	0	1

23.

A	B	C	$A+B$	$(A+B)C$
0	0	0	0	0
0	0	1	0	0
0	1	0	1	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	1
1	1	0	1	0
1	1	1	1	1

24.

A	B	C	$\overline{C}$	$A+B$	$A+B+\overline{C}$
0	0	0	1	0	1
0	0	1	0	0	0
0	1	0	1	1	1
0	1	1	0	1	1
1	0	0	1	1	1
1	0	1	0	1	1
1	1	0	1	1	1
1	1	1	0	1	1

25. No

A	B	AB	$\overline{AB}$	$\overline{A}$	$\overline{B}$	$\overline{A}\,\overline{B}$
0	0	0	1	1	1	1
0	1	0	1	1	0	0
1	0	0	1	0	1	0
1	1	1	0	0	0	0

26. Yes

A	B	$A+B$	$\overline{A+B}$	$\overline{A}$	$\overline{B}$	$\overline{A}\,\overline{B}$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

27. No

A	B	C	BC	$A+BC$	$A+B$	$(A+B)C$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	1	0
0	1	1	1	1	1	1
1	0	0	0	1	1	0
1	0	1	0	1	1	1
1	1	0	0	1	1	0
1	1	1	1	1	1	1

28. Yes

A	B	AB	$A+AB$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

29. Yes

A	B	C	$A+B$	$(A+B)+C$	$B+C$	$A+(B+C)$
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	1	1	1	1
0	1	1	1	1	1	1
1	0	0	1	1	0	1
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

30.

A	A	AA
0	0	0
1	1	1

AA is equivalent to A

31.

A	A	$A+A$
0	0	0
1	1	1

$A+A$ is equivalent to A

32.

A	0	$A+0$
0	0	0
1	0	1

$A+0$ is equivalent to A

33.

A	B	AB	$A+AB$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

$A+AB$ is equivalent to A

34.

A	B	$\overline{A}$	$\overline{A}+B$	$A(\overline{A}+B)$
0	0	1	1	0
0	1	1	1	0
1	0	0	0	0
1	1	0	1	1

$A(\overline{A}+B)$ is logically equivalent to AB

2.5 Laws of Logic

You may have noticed some common patterns running through some of the exercises by now. Let's examine those patterns in more detail.

First, let us look at the connections between the two sets of symbols we've used so far.

Logic	$p \wedge q$	$p \vee q$	$\sim p$	F	T
Boolean Algebra	AB	$A + B$	$\overline{A}$	0	1

In each case, we have symbols for negation, "or", and "and". There are also equivalences with False/True for logic, and 0/1 (off/on) for Boolean algebra and logic circuits. Let's see what else they have in common.

2.5.1 Identity Laws

Examining logical symbols first, let's fill in the following truth table.

p	0	1	$p \wedge 0$	$p \vee 0$	$p \wedge 1$	$p \vee 1$
0	0	1	0	0	0	1
1	0	1	0	1	1	1

From this table, we can see that

$$p \wedge 0 \Leftrightarrow 0$$

$$p \vee 0 \Leftrightarrow p$$

$$p \wedge 1 \Leftrightarrow p$$

$$p \vee 1 \Leftrightarrow 1$$

These are the identity laws, true for any proposition p. Notice that if we replaced all of the logic symbols in the table with Boolean algebra notation, we'd get

$$A \cdot 0 = 0$$
$$A + 0 = A$$
$$A \cdot 1 = A$$
$$A + 1 = 1$$

2.5.2 Idempotent Laws

Similarly, let's examine the following truth table.

p	$p \wedge p$	$p \vee p$
0	0	0
1	1	1

From this table, we can see that

$$p \wedge p \Leftrightarrow p$$
$$p \vee p \Leftrightarrow p$$

These are called the **idempotent** laws. Notice that if we replaced all of the logic symbols in the table with the equivalent set symbols and also by Boolean algebra notation, we'd get

$$A \cdot A = A$$

$$A + A = A$$

2.5.3 Complement Laws

If we wished, we could construct another truth table to show that

$$\sim(\sim p) \Leftrightarrow p$$

$$p \wedge \sim p \Leftrightarrow 0$$

$$p \vee \sim p \Leftrightarrow 1$$

These are called the **complement** laws. Notice that if we replaced all of the logic symbols in the table with Boolean algebra notation, we'd get

$$\overline{\overline{A}} = A$$

$$A \cdot \overline{A} = 0$$

$$A + \overline{A} = 1$$

2.5.4 Commutative Laws

Similarly,

$$p \wedge q \Leftrightarrow q \wedge p$$

$$p \vee q \Leftrightarrow q \vee p$$

These are called the **commutative** laws. Notice that if we replaced all of the logic symbols in the table with Boolean algebra notation, we'd get

$$AB = BA$$

$$A + B = B + A$$

2.5.5 Associative Laws

Similarly, we could construct another truth table to find that

$$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$$

$$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$$

These are called the **associative** laws. Notice that if we replaced all of the logic symbols in the table by Boolean algebra notation, we'd get

$$(AB)C = A(BC)$$

$$(A + B) + C = A + (B + C)$$

2.5.6 Summary

We can then summarize these laws as follows.

Law	Logic	Boolean Algebra
Identity	$p \wedge 1 \Leftrightarrow p$	$A \cdot 1 = A$
	$p \vee 1 \Leftrightarrow 1$	$A + 1 = 1$
	$p \wedge 0 \Leftrightarrow 0$	$A \cdot 0 = 0$
	$p \vee 0 \Leftrightarrow p$	$A + 0 = A$
Idempotent	$p \wedge p \Leftrightarrow p$	$AA = A$
	$p \vee p \Leftrightarrow p$	$A + A = A$
Complement	$\sim(\sim p) \Leftrightarrow p$	$\overline{\overline{A}} = A$
	$p \wedge \sim p \Leftrightarrow 0$	$A\overline{A} = 0$
	$p \vee \sim p \Leftrightarrow 1$	$A + \overline{A} = 1$
Commutative	$p \wedge q \Leftrightarrow q \wedge p$	$AB = BA$
	$p \vee q \Leftrightarrow q \vee p$	$A + B = B + A$
Associative	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	$(AB)C = A(BC)$
	$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	$(A + B) + C = A + (B + C)$

How, then, can we use these laws?

2.5.7 Simplifying Logical Expressions

We can now use these laws to simplify logical expressions or to prove logical equivalence without resorting to truth tables.

Suppose we wish to simplify $(p \wedge 1) \vee (\sim q \wedge 0) \vee (\sim r \wedge r)$. Note that this would require a truth table with 8 rows to show all combinations of p, q, and r. However, to do so using the laws of logic will require fewer steps.

The procedure for simplifying an expression using the laws of logic is to simplify each piece of the expression using a single law, then write the name of the law you are using to one side (writing the name of the law is **required**, and **not optional!**). If you are using more than one law, then use a separate line for each law/step.

Simplifying $(p \wedge 1) \vee (\sim q \wedge 0) \vee (\sim r \wedge r)$ would then give

$(p \wedge 1) \vee (\sim q \wedge 0) \vee (\sim r \wedge r)$	
$p \vee 0 \vee (\sim r \wedge r)$	identity
$(p \vee 0) \vee (\sim r \wedge r)$	associative
$p \vee (\sim r \wedge r)$	identity
$p \vee (r \wedge \sim r)$	commutative
$p \vee 0$	complement
p	identity

Our conclusion is therefore that $(p \wedge 1) \vee (\sim q \wedge 0) \vee (\sim r \wedge r) \Leftrightarrow p$.

We could also do an alternate solution, using a different order of steps to get our answer.

$(p \wedge 1) \vee (\sim q \wedge 0) \vee (\sim r \wedge r)$	
$p \vee 0 \vee (\sim r \wedge r)$	identity
$p \vee 0 \vee 0$	complement
$p \vee 0$	definition of "or"
p	identity

And we reach the same conclusion.

Example: Simplify $(p \vee \sim p) \wedge (\sim p \vee \sim p)$.

Answer:

$(p \vee \sim p) \wedge (\sim p \vee \sim p)$	
$1 \wedge (\sim p \vee \sim p)$	complement
$1 \wedge (\sim p)$	idempotent
$\sim p$	identity

(And if you applied the laws correctly but in a different order or combination, you should still come to the same, correct conclusion.)

Exercises for Section 2.5

1. Which of the following statements is always true?
   (a) Darth Vader is both evil and not evil.
   (b) Darth Vader is both evil and evil.
   (c) Darth Vader is either evil or evil.
   (d) Darth Vader is either evil or not evil.
2. Which of the following statements is always false?
   (a) The roadrunner has escaped from the wily coyote and he has not escaped from the wily coyote.
   (b) The roadrunner has escaped from the wily coyote and he has escaped from the wily coyote.
   (c) The roadrunner has escaped from the wily coyote or he has not escaped from the wily coyote.
   (d) The roadrunner has escaped from the wily coyote or he has escaped from the wily coyote.
3. Use a truth table to prove that the two idempotent laws are true.
4. Use a truth table to prove that the four identity laws are true.

Name the law of logic used in the following. Note that the variables have changed, but that the law is still valid.

5. $\sim q \vee 1 \Leftrightarrow 1$
6. $\overline{\overline{B}} = B$
7. $\sim r \wedge r \Leftrightarrow 0$
8. $\sim q \vee 0 \Leftrightarrow \sim q$
9. $\overline{B} \cdot 1 = \overline{B}$
10. $q \vee q \Leftrightarrow q$
11. $AB + \overline{AB} = 1$
12. $(\sim p \wedge q) \wedge \sim q \Leftrightarrow \sim p \wedge (q \wedge \sim q)$

Simplify the given expression, and state the name of the law you used. You should be able to do these in one step.

13. $r \vee 0$
14. $C + \overline{C}$
15. $\sim(\sim r)$
16. $\overline{A} + \overline{A}$
17. $\overline{B} \cdot 1$

Use the laws of logic to simplify the following logical expressions. If you're completely stuck, try using a truth table instead.

18. $(p \wedge p) \vee (q \wedge \sim q)$
19. $(p \vee p) \wedge (q \vee 0)$
20. $p \vee (q \wedge \sim q)$

Use the laws of logic to simplify the following Boolean expressions. If you're completely stuck, try using a truth table instead.

21. $(A + A)(B + \overline{B})$
22. $B \cdot 0 + AA$
23. $(B + \overline{B})(A + 1)$
24. $AB\overline{B}$

Prove the following Boolean expressions are equivalent using the laws of logic. If you're completely stuck, try using a truth table.

25. $(A\overline{A})\overline{B} = A(B\overline{B})$
26. $B \cdot 1 + A\overline{A} = \overline{\overline{B} \cdot 1}$
27. $(A + 0)(B + \overline{B}) = A$
28. $AA + \overline{B}\ \overline{B} = A + \overline{B}$

Answers to Section 2.5 Exercises

1. (d) is true because in logical symbols, $p \vee \sim p \Leftrightarrow 1$.

2. (a) is false because $p \wedge \sim p \Leftrightarrow 0$.

3. The two idempotent laws are true because the last column in each table is the same as for p.

p	p	$p \vee p$
0	0	0
1	1	1

p	p	$p \wedge p$
0	0	0
1	1	1

4. The four identity laws are true because the $p \wedge 0$ column is the same as 0, the $p \vee 0$ and $p \wedge 1$ columns are the same as p, and the $p \vee 1$ column is the same as 1.

p	0	1	$p \wedge 0$	$p \vee 0$	$p \wedge 1$	$p \vee 1$
0	0	1	0	0	0	1
1	0	1	0	1	1	1

5. identity

6. complement

7. complement

8. identity

9. identity

10. idempotent

11. complement

12. associative

13. r, using the identity law

14. 1, complement

15. r, complement

16. $\overline{A}$, idempotent

17. $\overline{B}$, identity

Note: for the following questions, there may be several different ways to get to the simplest answer. Also, you may take steps in a different order. If you are concerned about a different solution, please show your instructor. (Also, I haven't explicitly written out any steps involving either the Commutative or Associative laws.)

18. $(p \wedge p) \vee (q \wedge \sim q) \Leftrightarrow p \vee (q \wedge \sim q)$ Idempotent
$\Leftrightarrow p \vee 0$ Complement
$\Leftrightarrow p$ Identity

19. $(p \vee p) \wedge (q \vee 0) \Leftrightarrow p \wedge (q \vee 0)$ Idempotent
$\Leftrightarrow p \wedge q$ Identity

20. $p \vee (q \wedge \sim q) \Leftrightarrow p \vee 0$ Complement
$\Leftrightarrow p$ Identity

21. $(A + A)(B + \overline{B}) = A(B + \overline{B})$ Idempotent
$= A \cdot 1$ Complement
$= A$ Identity

22. $B \cdot 0 + AA = 0 + AA$ Identity
$= 0 + A$ Idempotent
$= A$ Identity

23. $(B + \overline{B})(A + 1) = 1 \cdot (A + 1)$ Complement
$= 1 \cdot 1$ Identity
$= 1$ Definition of "and"

24. $AB\overline{B} = A \cdot 0$ Complement
$= 0$ Identity

25. $(A\overline{A})\overline{B} = A(B\overline{B})$
$0 \cdot \overline{B} = A(B\overline{B})$ Complement
$0 \cdot \overline{B} = A \cdot 0$ Complement
$0 = A \cdot 0$ Identity
$0 = 0$ Identity

26.
$$
\begin{aligned}
B \cdot 1 + A\overline{A} &= \overline{\overline{B} \cdot 1} & \\
B + A\overline{A} &= \overline{\overline{B}} & \text{Identity} \\
B + 0 &= \overline{\overline{B}} & \text{Complement} \\
B &= \overline{\overline{B}} & \text{Identity} \\
B &= B & \text{Complement}
\end{aligned}
$$

27.
$$
\begin{aligned}
(A + 0)(B + \overline{B}) &= A & \\
A(B + \overline{B}) &= A & \text{Identity} \\
A \cdot 1 &= A & \text{Complement} \\
A &= A & \text{Identity}
\end{aligned}
$$

28.
$$
\begin{aligned}
AA + \overline{B}\ \overline{B} &= A + \overline{B} & \\
A + \overline{B} &= A + \overline{B} & \text{Idempotent}
\end{aligned}
$$

2.6 More Laws of Logic

2.6.1 De Morgan's Laws

Let's examine the following truth table.

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

From this table, we can see that

$$\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$$

We can draw a similar table to show that

$$\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$$

These are called **De Morgan's laws**. Notice that if we replaced all of the logic symbols in the table by Boolean algebra notation, we'd get

$$\begin{aligned} \overline{AB} &= \overline{A} + \overline{B} \\ \overline{A + B} &= \overline{A}\ \overline{B} \end{aligned}$$

2.6.2 Distributive Laws

Similarly, we could use a truth table to show that

$$\begin{aligned} p \wedge (q \vee r) &\Leftrightarrow (p \wedge q) \vee (p \wedge r) \\ p \vee (q \wedge r) &\Leftrightarrow (p \vee q) \wedge (p \vee r) \end{aligned}$$

These are called the **distributive** laws. Notice that if we replaced all of the logic symbols in the table with Boolean algebra notation, we'd get

$$\begin{aligned} A(B + C) &= AB + AC \\ A + BC &= (A + B)(A + C) \end{aligned}$$

2.6.3 Absorption Laws

Similarly,

$$p \wedge (p \vee q) \Leftrightarrow p$$
$$p \wedge (\sim p \vee q) \Leftrightarrow p \wedge q$$

We can draw a similar table to show that

$$p \vee (p \wedge q) \Leftrightarrow p$$
$$p \vee (\sim p \wedge q) \Leftrightarrow p \vee q$$

These are called the **absorption** laws. Notice that if we replaced all of the logic symbols in the table with the equivalent set symbols and also by Boolean algebra notation, we'd get

$$A(A + B) = A$$
$$A(\overline{A} + B) = AB$$
$$A + AB = A$$
$$A + \overline{A}B = A + B$$

2.6.4 Summary

We can then summarize these laws as follows.

Law	Logic	Boolean Algebra
Identity	$p \wedge 1 \Leftrightarrow p$ $p \vee 1 \Leftrightarrow 1$ $p \wedge 0 \Leftrightarrow 0$ $p \vee 0 \Leftrightarrow p$	$A \cdot 1 = A$ $A + 1 = 1$ $A \cdot 0 = 0$ $A + 0 = A$
Idempotent	$p \wedge p \Leftrightarrow p$ $p \vee p \Leftrightarrow p$	$AA = A$ $A + A = A$
Complement	$\sim(\sim p) \Leftrightarrow p$ $p \wedge \sim p \Leftrightarrow 0$ $p \vee \sim p \Leftrightarrow 1$	$\overline{\overline{A}} = A$ $A\overline{A} = 0$ $A + \overline{A} = 1$
Commutative	$p \wedge q \Leftrightarrow q \wedge p$ $p \vee q \Leftrightarrow q \vee p$	$AB = BA$ $A + B = B + A$
Associative	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$ $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	$(AB)C = A(BC)$ $(A + B) + C = A + (B + C)$
De Morgan's	$\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$ $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$	$\overline{AB} = \overline{A} + \overline{B}$ $\overline{A + B} = \overline{A}\ \overline{B}$
Distributive	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	$A(B + C) = AB + AC$ $A + BC = (A + B)(A + C)$
Absorption	$p \wedge (p \vee q) \Leftrightarrow p$ $p \wedge (\sim p \vee q) \Leftrightarrow p \wedge q$ $p \vee (p \wedge q) \Leftrightarrow p$ $p \vee (\sim p \wedge q) \Leftrightarrow p \vee q$	$A(A + B) = A$ $A(\overline{A} + B) = AB$ $A + AB = A$ $A + \overline{A}B = A + B$

2.6.5 Simplifying Logical Expressions

The real power of these laws lies in simplifying logical expressions and in proofs. (Remember – one law per line, must write name of law!)

Example: Simplify $(p \wedge q) \vee (p \wedge \sim q)$.

Answer:

$$
\begin{array}{cl}
(p \wedge q) \vee (p \wedge \sim q) & \\
p \wedge (q \vee \sim q) & \text{distributive} \\
p \wedge 1 & \text{complement} \\
p & \text{identity}
\end{array}
$$

Example: Simplify $AB(\overline{A} + \overline{B})$.

Answer:

$$
\begin{array}{cl}
AB(\overline{A} + \overline{B}) & \\
AB\overline{A} + AB\overline{B} & \text{distributive} \\
A\overline{A}B + AB\overline{B} & \text{commutative} \\
0 \cdot B + A \cdot 0 & \text{complement} \\
0 + 0 & \text{identity} \\
0 & \text{definition of “or”}
\end{array}
$$

Note that for many of these exercises, there is more than one way to answer. Another equally valid simplification looks like the following.

$$
\begin{array}{cl}
AB(\overline{A} + \overline{B}) & \\
AB\overline{AB} & \text{De Morgan's} \\
0 & \text{complement}
\end{array}
$$

This is a much shorter answer, but does require a flash of insight at the $(\overline{A} + \overline{B})$ pattern.

2.6.6 Proofs

Example: Show that $\overline{A + \overline{A}\ \overline{B}} = \overline{A}B$.

Answer:

Let's examine the left-hand side.

$$
\begin{array}{rcll}
\overline{A + \overline{A}\ \overline{B}} & = & \overline{A}B & \\
\overline{A + \overline{B}} & = & \overline{A}B & \text{absorption} \\
\overline{A}\ \overline{\overline{B}} & = & \overline{A}B & \text{De Morgan's} \\
\overline{A}B & = & \overline{A}B & \text{complement}
\end{array}
$$

and the fact that the left-hand side is equivalent to the right-hand side completes our proof.

Example: Show that $(A+\overline{C})(\overline{C}+AB) = AB+\overline{C}$.

Answer:

Let's examine the left-hand side.

$$
\begin{aligned}
(A+\overline{C})(\overline{C}+AB) &= AB+\overline{C} & \\
(\overline{C}+A)(\overline{C}+AB) &= AB+\overline{C} & \text{commutative} \\
\overline{C}+A(AB) &= AB+\overline{C} & \text{distributive} \\
\overline{C}+(AA)B &= AB+\overline{C} & \text{associative} \\
\overline{C}+AB &= AB+\overline{C} & \text{idempotent} \\
AB+\overline{C} &= AB+\overline{C} & \text{commutative}
\end{aligned}
$$

QED. (QED is short for the Latin phrase "quod erat demonstrandum", which means "it has been demonstrated".)

Exercises for Section 2.6

(Note that these are the same exercises as at the beginning of section 1.5, but with a little twist.) Let p be "Rich is seven feet tall" and q be "Susan has brown hair." Translate the following English sentences into logical notation. Then, use one of the laws of logic to write an equivalent logical expression. Finally, translate your new expression back into an English sentence.

1. Rich is seven feet tall or he is seven feet tall.
2. Susan has brown hair and she has brown hair.
3. Either Rich is not seven feet tall or Susan does not have brown hair.
4. It is not true that Rich is seven feet tall and Susan has brown hair.
5. It is not true that Rich is seven feet tall or Susan has brown hair.
6. Rich is not seven feet tall and Susan does not have brown hair.
7. Rich is seven feet tall and Susan has brown hair.
8. Susan has brown hair or Rich is seven feet tall.

Name the law of logic used in the following. Note that the variables have changed, but that the law is still valid.

9. $\sim(q \vee r) \Leftrightarrow \sim q \wedge \sim r$
10. $\overline{B}(B+\overline{A}) = \overline{B}\ \overline{A}$
11. $(p \wedge q) \vee (p \wedge \sim q) \Leftrightarrow p \wedge (q \vee \sim q)$
12. $\overline{\overline{A}+C} = A\overline{C}$
13. $B + A\overline{C} = (B+A)(B+\overline{C})$
14. $\sim p \vee (p \wedge r) \Leftrightarrow \sim p \vee r$

Simplify the given expression, and state the name of the law you used. You should be able to do these in a single step.

15. $\overline{A} + A\overline{B}$
16. $\overline{AB} + \overline{AB}$
17. $(A+B)(B+C)$
18. $q \vee (q \wedge r)$

19. $\overline{C} + C$

20. $\overline{\overline{A}\;\overline{B}}$

For the following exercises, let p be "The moon is made of green cheese" and q be "The earth is made of green cheese." Translate the following English sentences into logical notation. Then, use one of the laws of logic to write an equivalent logical expression. Finally, translate your new expression back into an English sentence. (Note that these are the same exercises as at the beginning of section 2.1, but with a little twist.)

21. Either the moon is made of green cheese or both the moon and the earth are made of green cheese.

22. The earth is made of green cheese and either the earth or the moon is made of green cheese.

23. Either the earth is made of green cheese while the moon is not, or the moon is made of green cheese.

24. The earth is made of green cheese and either the moon is made of green cheese or the earth is not.

25. Remembering that $\oplus$ is "exclusive or", show that $\mathrm{A}\oplus\mathrm{B} = \overline{A}\mathrm{B} + \mathrm{A}\overline{B}$ by using a truth table.

26. The `NAND` gate (not-AND) has the following truth table. Use DeMorgan's laws to find an equivalent Boolean expression using only OR and NOT, and show that your expression has the same truth table.

A	B	A `NAND` $B = \overline{AB}$
0	0	1
0	1	1
1	0	1
1	1	0

Simplify the following Boolean expressions using the laws of logic. If you're stuck, try using a truth table.

27. $A + \overline{C} + B + \overline{A} + \overline{B}$

28. $A + \overline{B} + A + B + A$

29. $\overline{\overline{A}\;\overline{B}}$

30. $\overline{\overline{A}+\overline{B}}$
31. $\overline{A}+B+A\overline{B}$
32. $A\overline{B}\,\overline{C}+A\overline{B}C$
33. $\overline{A}BC+\overline{A}B\overline{C}+\overline{A}B\overline{D}+\overline{A}BD$
34. $AB+A+\overline{AB}$
35. $A+\overline{B}CD+\overline{B}$
36. $\overline{A}\,\overline{B}(A+B)$
37. $(\overline{A}+\overline{B})(A+B)$
38. $A+\overline{A}B+\overline{B}C$
39. $B(A+C)+\overline{A}B\overline{C}$
40. $(A+B+C)(A+B+\overline{C})$

Prove that the following Boolean expressions are equivalent by using the laws of logic. If you're stuck, try using a truth table.

41. $B\overline{B}+AA=A$
42. $\overline{A}(B+\overline{B})=\overline{A}$
43. $ABC+AB\overline{C}=AB$
44. $AB+\overline{AB}C=AB+C$
45. $A+AB+ABC=A$
46. $\overline{A}C+A\overline{B}C=\overline{A}C+\overline{B}C$
47. $\overline{AB}(A+B)=\overline{A}B+A\overline{B}$
48. $\overline{\overline{\overline{A}BC}+D}=\overline{A}BC\overline{D}$
49. $A\,\overline{B}\,\overline{\overline{A}\,\overline{C}}=A\,\overline{B}$

Answers to Section 2.6 Exercises

1. $p \vee p \Leftrightarrow p$. Rich is seven feet tall.
2. $q \wedge q \Leftrightarrow q$. Susan has brown hair.
3. $\sim p \vee \sim q \Leftrightarrow \sim(p \wedge q)$. It is not the case that Rich is seven feet tall and Susan has brown hair.
4. $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$. Rich is not seven feet tall or Susan does not have brown hair.
5. $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$. Rich is not seven feet tall and Susan does not have brown hair.
6. $\sim p \wedge \sim q \Leftrightarrow \sim(p \vee q)$. It is not the case that Rich is seven feet tall or Susan has brown hair.
7. $p \wedge q \Leftrightarrow q \wedge p$. Susan has brown hair and Rich is seven feet tall.
8. $q \vee p \Leftrightarrow p \vee q$. Rich is seven feet tall or Susan has brown hair.
9. De Morgan's
10. absorption
11. distributive
12. De Morgan's
13. distributive
14. absorption
15. $\overline{A} + \overline{B}$, absorption
16. $\overline{AB}$, idempotent
17. $B + AC$, distributive
18. q, absorption
19. 1, complement
20. $A + B$, De Morgan's
21. $p \vee (p \wedge q) \Leftrightarrow p$. The moon is made of green cheese.
22. $q \wedge (q \vee p) \Leftrightarrow q$. The earth is made of green cheese.

23. $(q \wedge {\sim}p) \vee p \Leftrightarrow p \vee ({\sim}p \wedge q) \Leftrightarrow p \vee q$. (note: I'm using the commutative laws to rearrange things) The moon or the earth is made of green cheese.

24. $q \wedge (p \vee {\sim}q) \Leftrightarrow q \wedge p$. The earth and the moon are made of green cheese.

25.

A	B	$A \oplus B$	$\overline{A}$	$\overline{B}$	$\overline{A}B$	$A\overline{B}$	$\overline{A}B + A\overline{B}$
0	0	0	1	1	0	0	0
0	1	1	1	0	1	0	1
1	0	1	0	1	0	1	1
1	1	0	0	0	0	0	0

26. By DeMorgan's law, $\overline{AB} = \overline{A} + \overline{B}$

A	B	A NAND $B = \overline{AB}$	$\overline{A}$	$\overline{B}$	$\overline{A} + \overline{B}$
0	0	1	1	1	1
0	1	1	1	0	1
1	0	1	0	1	1
1	1	0	0	0	0

27. 1

28. 1

29. $A + B$

30. AB

31. 1

32. $A\overline{B}$

33. $\overline{A}B$

34. 1

35. $A + \overline{B}$

36. 0

37. $A\overline{B} + \overline{A}B$

38. $A + B + C$

39. B

40. $A + B$

41.
$$\begin{aligned} B\overline{B} + AA &= A \\ 0 + AA &= A && \text{complement} \\ AA &= A && \text{identity} \\ A &= A && \text{idempotent} \end{aligned}$$

42.
$$\begin{aligned} \overline{A}(B + \overline{B}) &= \overline{A} \\ \overline{A}(1) &= \overline{A} && \text{complement} \\ \overline{A} &= \overline{A} && \text{identity} \end{aligned}$$

43.
$$\begin{aligned} ABC + AB\overline{C} &= AB \\ AB(C + \overline{C}) &= AB && \text{distributive} \\ AB(1) &= AB && \text{complement} \\ AB &= AB && \text{identity} \end{aligned}$$

44.
$$\begin{aligned} AB + \overline{AB}C &= AB + C \\ (AB) + (\overline{AB})C &= AB + C && \text{associative (can skip this step)} \\ AB + C &= AB + C && \text{absorption} \end{aligned}$$

45.
$$\begin{aligned} A + AB + ABC &= A \\ A + ABC &= A && \text{absorption} \\ A + A(BC) &= A && \text{associative (can skip)} \\ A &= A && \text{absorption} \end{aligned}$$

46.
$$\begin{aligned} \overline{A}C + A\overline{B}C &= \overline{A}C + \overline{B}C \\ (\overline{A} + A\overline{B})C &= \overline{A}C + \overline{B}C && \text{distributive} \\ (\overline{A} + \overline{B})C &= \overline{A}C + \overline{B}C && \text{absorption} \\ \overline{A}C + \overline{B}C &= \overline{A}C + \overline{B}C && \text{distributive} \end{aligned}$$

47.

$$\begin{aligned} \overline{AB}(A+B) &= \overline{A}B + A\overline{B} & \\ (\overline{A}+\overline{B})(A+B) &= \overline{A}B + A\overline{B} & \text{De Morgan's} \\ \overline{A}A + \overline{A}B + \overline{B}A + \overline{B}B &= \overline{A}B + A\overline{B} & \text{distributive} \\ 0 + \overline{A}B + \overline{B}A + 0 &= \overline{A}B + A\overline{B} & \text{complement} \\ \overline{A}B + A\overline{B} &= \overline{A}B + A\overline{B} & \text{identity} \end{aligned}$$

48.

$$\begin{aligned} \overline{\overline{\overline{A}BC} + D} &= \overline{A}BC\overline{D} & \\ \overline{\overline{\overline{A}BC}}\;\overline{D} &= \overline{A}BC\overline{D} & \text{De Morgan's} \\ \overline{A}BC\overline{D} &= \overline{A}BC\overline{D} & \text{complement} \end{aligned}$$

49.

$$\begin{aligned} A\overline{B}\;\overline{\overline{A}\;\overline{C}} &= A\overline{B} & \\ A\overline{B}(A+C) &= A\overline{B} & \text{De Morgan's} \\ A\overline{B}A + A\overline{B}C &= A\overline{B} & \text{distributive} \\ A\overline{B} + A\overline{B}C &= A\overline{B} & \text{idempotent} \\ A\overline{B} &= A\overline{B} & \text{absorption} \end{aligned}$$

2.7 The Conditional

2.7.1 The Conditional Connective

Suppose we have two propositions, p and q. Remembering that connectives are operations which join two or more propositions (like "and" and "or"), the conditional connective is

If p, then q.

In symbols, this is written as $p \to q$, and when read aloud, you say "p implies q". When using the conditional, the first proposition is called the **hypothesis** and the second is called the **conclusion**.

There are other ways to state the conditional. "If p, then q" is equivalent to

(a) p implies q

(b) q, if p

(c) p is sufficient for q

(d) q is necessary for p

(e) p only if q

We'll only be using the "If p, then q" and "p implies q" conventions in this course.

But what does the conditional mean? Suppose you have an insurance contract[3] which reads:

> If your house burns down, then the insurance company will give you \$1 000 000.

Let us further suppose your house burns down. Under the contract, the insurance company must give you one million dollars. If it doesn't, the contract has been violated. But if your house doesn't burn down and the company doesn't give you any money, the contract still holds. If your house doesn't burn down and out of boundless generosity the company give you one million dollars anyway, the contract still holds. The only circumstances under which the contract is violated is when your house does burn down but the company fails to give you one million dollars. This leads to the following truth table.

[3] My thanks go to my colleague Gilles Cazelais for providing the idea for this example.

House burns down?	You get \$1 000 000?	The contract holds
no	no	yes
no	yes	yes
yes	no	no
yes	yes	yes

To generalize to the propositions p and q,

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

What does this mean? It means that if p is true and q is false, then the implication $p \to q$ cannot also be true. It also means that if $p \to q$ is true, then you cannot have p true and q false at the same time.

Let's look at another example. Suppose that the following conditional is true: "If Barney is a dog, then Barney has four legs." This means that if the first proposition, "Barney is a dog," is true, then only one conclusion may be reached, that the second is true and Barney has four legs. However, if p is false and Barney is not a dog, then our conditional doesn't have anything to say about the number of legs Barney may have. If Barney is not a dog (Barney is a snake, octopus, bug, person, or pond), then Barney may not have four legs. If Barney is not a dog (Barney is a cat, giraffe, woolly mammoth, or table), Barney may have four legs. Our conditional does not deal with what you may conclude when p is false.

Example: Suppose that the statement "If Pat sleeps in, she will be late for class" is true. Answer the following questions.

(a) Pat sleeps in. Is she late for class?

(b) Pat does not sleep in. Is she late for class?

(c) Pat is late for class. Did she sleep in?

(d) Pat is not late for class. Did she sleep in?

Answer:

(a) Yes.

(b) Maybe. (Perhaps she ran into traffic or was eaten by bears. Remember that the conditional has nothing to say when the first proposition is false.)

(c) Maybe. (Again, maybe there was another reason for her lateness.)

(d) No.

2.7.2 The Converse

If $p \to q$ is true, is it also true that $q \to p$? If $p \to q$ is called the conditional, then $q \to p$ is called the **converse**. Let's use our previous example again, which was "If Barney is a dog, then Barney has four legs." If this statement is true, is it also true that "If Barney has four legs, then Barney is a dog"?

Clearly this second statement is not also true, since Barney could be a four-legged creature that is not a dog, such as a cat, mouse, grizzly bear, or mountain goat.

Let's write out the truth table for the conditional and the converse.

p	q	$p \to q$	$q \to p$
F	F	T	T
F	T	T	F
T	F	F	T
T	T	T	T

Here's how to fill in the columns for any logical expression containing the conditional connective ($\to$). Let's call the propositions "first" and "second" so we don't get confused with p and q. For the conditional first $\to$ second, it will be true for all cases **except** when the first is true and the second is false. So for $q \to p$, look for the row in which q is true (rows 2 and 4) and p is false (1 and 2). Then q is true and p is false only for row 2. Therefore, all rows except for the second get True and the second row gets False. So you can also see from the truth table that the conditional $p \to q$ and the converse $q \to p$ are **not** logically equivalent.

2.7.3 The Contrapositive

What about the contrapositive, $\sim q \rightarrow \sim p$? For our familiar example, that would be asking whether "If Barney is a dog, then Barney has four legs" is equivalent to "If Barney does not have four legs, then Barney is not a dog". This at least looks a little more promising. Let's try the truth table, but this time using 1s and 0s instead of True and False.

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	0	0	1	1

Remember, you'll fill in all 1s except for the row where the first is true and the second is false. So look for the rows with $\sim q$ true (rows 1 and 3) and $\sim p$ false (rows 3 and 4). So the row with the zero will be the third row. As the last two columns are identical, then the conditional $p \rightarrow q$ is equivalent to the contrapositive $\sim q \rightarrow \sim p$.

> **Example:** Write the contrapositive for the statement, "If today is sunny, Pat will work in the garden."
>
> Answer: To get the contrapositive, negate each proposition and then reverse the order. So the contrapositive is "If Pat is not working in the garden, then today is not sunny."

2.7.4 The Inverse

If $p \rightarrow q$ is the conditional, then the proposition $\sim p \rightarrow \sim q$ is called the inverse. So if the conditional is "If Barney is a dog, then Barney has four legs", then the inverse of that would be "If Barney is not a dog, then Barney does not have four legs." You can see directly from this example that the conditional and the inverse are **not** equivalent!

Here's what the truth table looks like. (Remember that the $\rightarrow$ means that the value is 1 except when the first is true and the second is false.)

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim p \to \sim q$
0	0	1	1	1	1
0	1	1	0	1	0
1	0	0	1	0	1
1	1	0	0	1	1

Example: Draw the truth tables for the conditional ($p \to q$), the converse ($q \to p$), the inverse ($\sim p \to \sim q$), and the contrapositive ($\sim q \to \sim p$). Are any of these propositions logically equivalent?

Answer: Here's the big truth table.

p	q	$\sim p$	$\sim q$	$p \to q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
0	0	1	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	1	0	0	1	1	1	1

Since the 5th and 8th columns are identical, the conditional ($p \to q$) is logically equivalent to the contrapositive ($\sim q \to \sim p$). Since the 6th and 7th columns are identical, the converse ($q \to p$) is logically equivalent to the inverse ($\sim p \to \sim q$).

2.7.5 The "or" form of the conditional

Can the conditional $p \to q$ be rewritten using our basic connectives "and", "or", and "not"? Yes, it can, because you can see by the truth table below that $p \to q$ is logically equivalent to $\sim p \vee q$.

p	q	$\sim p$	$\sim p \vee q$	$p \to q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

This means that "If Barney is a dog, then Barney has four legs" is logically equivalent to "Either Barney is not a dog or he has four legs."

Example: Consider the conditional $p \rightarrow q$. Is the converse $(q \rightarrow p)$ logically equivalent to $\sim p \vee q$, $\sim p \wedge q$, $p \vee \sim q$, or $p \wedge \sim q$?

Answer: Let's write in the truth table and compare columns.

p	q	$\sim p$	$\sim q$	$q \rightarrow p$	$\sim p \vee q$	$\sim p \wedge q$	$p \vee \sim q$	$p \wedge \sim q$
0	0	1	1	1	1	0	1	0
0	1	1	0	0	1	1	0	0
1	0	0	1	1	0	0	1	1
1	1	0	0	1	1	0	1	0

As can be seen from the table, the columns for $q \rightarrow p$ and $p \vee \sim q$ are identical, so these two expressions are logically equivalent.

2.7.6 De Morgan's Law and the Contrapositive

Consider the conditional $(p \wedge q) \rightarrow r$. The contrapositive would be $\sim r \rightarrow \sim(p \wedge q)$. Applying De Morgan's Law gives $\sim r \rightarrow (\sim p \vee \sim q)$. Notice the change from the "and" in the conditional to the "or" in the modified contrapositive. Forgetting to make that change is an easy trap to fall into.

Example: Consider the conditional "If Pat sleeps in or runs into traffic, she will be late for class." What is the contrapositive? Use De Morgan's Law to find your answer.

Answer: The conditional here is

$$\text{(sleep or traffic)} \rightarrow \text{late}$$

The contrapositive is then

$$\begin{aligned} \sim\text{late} &\rightarrow \sim\text{(sleep or traffic)} \\ &\rightarrow \sim\text{sleep and} \sim\text{traffic} \end{aligned}$$

Therefore, the contrapositive is "If Pat is not late for class, then she didn't sleep in **and** did not run into traffic." The "or" changes into an "and" because of De Morgan's Law.

Exercises for Section 2.7

In the following exercises, let p denote "The movie was popular" and q denote "The movie will make a lot of money." Translate the following propositions into English sentences.

1. $p \to q$
2. $\sim p \to \sim q$
3. $\sim q \to \sim p$
4. $q \to p$
5. $\sim p \vee q$
6. $p \wedge \sim q$

In the following exercises, let p denote "Pat eats a burger for dinner" and q denote "Pat is too full for dessert." Translate the following sentences into logical symbols.

7. If Pat eats a burger for dinner, she will be too full for dessert.
8. If Pat does not eat a burger for dinner, she will not be too full for dessert.
9. If Pat is too full for dessert, then she ate a burger for dinner.
10. If Pat is not too full for dessert, then she did not eat a burger for dinner.
11. If Pat is too full for dessert, then she did not eat a burger for dinner.
12. Pat being too full for dessert implies that she ate a burger for dinner.
13. Pat not being too full for dessert implies that she did not eat a burger for dinner.
14. Pat not eating a burger for dinner implies that she will not be too full for dessert.
15. Pat eating a burger for dinner implies that she will be too full for dessert.
16. Either Pat does not eat a burger for dinner or she will be too full for dessert.
17. Either Pat is not too full for dessert or she ate a burger for dinner.

18. Either Pat is too full for dessert or she did not eat a burger for dinner.

19. The following conditional statement is true: If Pat is eaten by bears, she will not finish her marking. Given that, answer the following questions.

    (a) Pat is eaten by bears. Did she finish her marking?

    (b) Pat is not eaten by bears. Did she finish her marking?

    (c) Pat finished her marking. Was she eaten by bears?

    (d) Pat did not finish her marking. Was she eaten by bears?

20. The following conditional statement is true: If Rich is asleep, then he is not playing ping-pong. Given that, answer the following questions.

    (a) Rich is playing ping-pong. Is he asleep?

    (b) Rich is asleep. Is he playing ping-pong?

    (c) Rich is not asleep. Is he playing ping-pong?

    (d) Rich is not playing ping-pong. Is he asleep?

Of course, for the previous questions, I chose situations in which you can use common sense to determine the answer. However, the true test of whether you understand the concept is to replace the above propositions by complete nonsense.

21. The following conditional statement is true: If ettercaps are green, then toves are slithy. Given that, answer the following questions.

    (a) Toves are slithy. Are ettercaps green?

    (b) Toves are not slithy. Are ettercaps green?

    (c) Ettercaps are green. Are toves slithy?

    (d) Ettercaps are red. Are toves slithy?

22. The following conditional statement is true: If the hare reads the Times Colonist, the tortoise will take out the recycling. Given that, answer the following questions.

    (a) The hare does not read the Times Colonist. Will the tortoise take out the recycling?

(b) The hare reads the Times Colonist. Will the tortoise take out the recycling?

(c) The tortoise takes out the recycling. Does the hare read the Times Colonist?

(d) The tortoise is not taking out the recycling. Does the hare read the Times Colonist?

Given the conditional statement, "If frattling is non-responsive, then the runges must be strunking", write the corresponding English sentences for the following.

23. The contrapositive ($\sim q \to \sim p$)
24. The converse ($q \to p$)
25. The inverse ($\sim p \to \sim q$)
26. The "or" form ($\sim p \vee q$)
27. Given the conditional statement, "If Bossy is mooing, she must be a cow," which of the four following statements is the contrapositive ($\sim q \to \sim p$)?
    (a) If Bossy is not a cow, she is not mooing.
    (b) If Bossy is a cow, then she is mooing.
    (c) If Bossy is mooing, then she must be a cow.
    (d) If Bossy is not mooing, then she must not be a cow.
28. Given the conditional statement, "If Bossy is mooing, she must be a cow," which of the four following statements is the converse ($q \to p$)?
    (a) If Bossy is not a cow, she is not mooing.
    (b) If Bossy is a cow, then she is mooing.
    (c) If Bossy is mooing, then she must be a cow.
    (d) If Bossy is not mooing, then she must not be a cow.
29. If the statement "If Bossy is mooing, then she must be a cow," is a true statement, which of the four following statements is also true?
    (a) If Bossy is not a cow, she is not mooing.
    (b) If Bossy is a cow, then she is mooing.

(c) Either Bossy is mooing or she is a cow.

(d) If Bossy is not mooing, then she must not be a cow.

30. Which of the following is the correct "or" form for the conditional "If Bossy is mooing, then she must be a cow"?

    (a) Bossy is a cow or she is not mooing.

    (b) Bossy is not a cow or she is not mooing.

    (c) Bossy is not a cow or she is mooing.

    (d) Bossy is a cow or she is mooing.

31. If the statement "If Bossy is mooing, then she must be a cow" is a true statement, which of the following cannot occur?

    (a) Bossy is mooing and she is a cow.

    (b) Bossy is mooing and she is not a cow.

    (c) Bossy is not mooing and she is not a cow.

    (d) Bossy is not mooing and she is a cow.

32. Consider the following "or" form statement, "Either Superman has a cape or he cannot fly." Which of the following is the correct form of the corresponding conditional?

    (a) If Superman does not have a cape, then he cannot fly.

    (b) If Superman has a cape, then he can fly.

    (c) If Superman can fly, then he has a cape.

    (d) If Superman cannot fly, then he doesn't have a cape.

33. Consider the conditional "If John has the flu or misses the bus, he will be late for work". Which of the following is the corresponding contrapositive statement ($\sim q \rightarrow \sim p$)?

    (a) If John is late for work, then he had the flu or missed the bus.

    (b) If John is late for work, then he did not have the flu or did not miss the bus.

    (c) If John is not late for work, then he did not have the flu or did not miss the bus.

(d) If John is not late for work, then he did not have the flu and did not miss the bus.

34. Consider the conditional "If Rich doesn't show his work or makes a mistake, then he will not get full credit". Which of the following is the corresponding contrapositive statement ($\sim q \to \sim p$)?

(a) If Rich received full credit, then he showed his work and did not make a mistake.

(b) If Rich received full credit, then he showed his work or did not make a mistake.

(c) If Rich did not get full credit, then he didn't show his work and made a mistake.

(d) If Rich did not get full credit, then he didn't show his work or made a mistake.

35. Consider the conditional "If Pat is late and has not called her husband, he will be worried". Which of the following is the corresponding contrapositive statement ($\sim q \to \sim p$)?

(a) If Pat's husband is not worried, then she is not late and did call him.

(b) If Pat's husband is not worried, then she is not late or did call him.

(c) If Pat's husband is worried, then she is late and has not called him.

(d) If Pat's husband is not worried, then she is late and did not call him.

36. Consider the conditional "If grunkles are circular, then runges are square and triptrops are blue". Which of the following is the corresponding contrapositive statement ($\sim q \to \sim p$)?

(a) If runges are not square and triptrops are not blue, then grunkles are not circular.

(b) If runges are not square or triptrops are not blue, then grunkles are circular.

(c) If runges are not square or triptrops are not blue, then grunkles are not circular.

(d) If runges are not square and triptrops are not blue, then grunkles are circular.

Answers to Section 2.7 Exercises

1. If the movie was popular, then it will make a lot of money. (Or: The movie's popularity implies that it will make a lot of money.)
2. If the movie was not popular, then it will not make a lot of money.
3. If the movie did not make a lot of money, then the movie was not popular.
4. If the movie will make a lot of money, then it is popular.
5. The movie was not popular or it made a lot of money.
6. The movie was popular and it did not make a lot of money.
7. $p \to q$
8. $\sim p \to \sim q$
9. $q \to p$
10. $\sim q \to \sim p$
11. $q \to \sim p$
12. $q \to p$
13. $\sim q \to \sim p$
14. $\sim p \to \sim q$
15. $p \to q$
16. $\sim p \vee q$
17. $\sim q \vee p$
18. $q \vee \sim p$
19. a) No b) Maybe c) No d) Maybe
20. a) No b) No c) Maybe d) Maybe
21. a) Maybe b) No c) Yes d) Maybe
22. a) Maybe b) Yes c) Maybe d) No
23. If the runges are not strunking, then the frattling must be responsive.
24. If the runges are strunking, then the frattling is non-responsive.

25. If the frattling is responsive, then the runges must not be strunking.
26. The frattling is responsive or the runges are strunking.
27. (a)
28. (b)
29. (a)
30. (a)
31. (b)
32. If you let $p =$ "Superman does not have a cape", then the answer is (a). If instead you let $p =$ "Superman cannot fly", then the answer is (c). So either (a) or (c) would be correct.
33. (d)
34. (a)
35. (b)
36. (c)

2.8 The Biconditional

2.8.1 The Biconditional Connective

Consider the conditional "If you live in Victoria, then you live in BC." Remembering that the conditional has nothing to say if the first proposition is false, then it is possible for you to not live in Victoria but to still live in BC (Nanaimo, Vancouver, etc.). It is also possible for you to not live in Victoria and also not live in BC (Calgary, AB or Toronto, ON).

Let's now consider the conditional "If the temperature outside is below 0°C, then it is freezing outside." If I were to use this sentence in everyday English, I probably mean "If the temperature outside is below 0°C, then it is freezing outside **AND** if the temperature outside is **not** below 0°C, then it's **not** freezing outside." So we could probably do with a new proposition that means "If p, then q **and** if $\sim p$, then $\sim q$." This connective is called the **biconditional**, $p \leftrightarrow q$.

The truth table for the biconditional, then, is

p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

So if p and q have the same value, then the biconditional is true and otherwise it's false. (In a sense, then, it's the negation of "exclusive or", $p \oplus q$.) There are a number of ways to specify the biconditional in English:

(a) If and only if p, then q.

(b) p if and only if q.

(c) If p, then q, and vice versa.

(d) If p, then q, and if $\sim p$, then $\sim q$.

We'll mostly be using the first construction, using "if and only if".

> **Example:** Draw the truth tables for $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$. Are they logically equivalent? Also, are $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (\sim p \rightarrow \sim q)$ logically equivalent?

Answer: Let's draw a big truth table:

p	q	$p \leftrightarrow q$	$\sim p$	$\sim q$	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$(p \rightarrow q) \wedge (\sim p \rightarrow \sim q)$
0	0	1	1	1	1	1	1	1	1
0	1	0	1	0	1	0	0	0	0
1	0	0	0	1	0	1	1	0	0
1	1	1	0	0	1	1	1	1	1

So by looking at the relevant columns, we can see that all three logical expressions are equivalent. So ("if p, then q" AND "if q, then p") is equivalent to the biconditional.[4]

Example: Consider the following conditional statements.

(a) "If two lines are perpendicular, then the angle between them is 90°."

(b) "If a polygon is a right triangle, then it has three sides."

Which of these sentences would still be true if it were written in the form of the biconditional?

Answer:

(a) would still be true since if two lines are not perpendicular, then the angle between them is not 90°.

(b) would not be true, since there are many triangles that aren't right triangles.

Example: The following biconditional statement is true: "If and only if Pat finishes her marking, she will not feel guilty." Given that, answer the following questions.

(a) Pat feels guilty. Did she finish her marking?

(b) Pat does not feel guilty. Did she finish her marking?

(c) Pat finished her marking. Does she feel guilty?

(d) Pat did not finish her marking. Does she feel guilty?

[4]We could have also noted that since the converse $(q \rightarrow p)$ is logically equivalent to the inverse $(\sim p \rightarrow \sim q)$, then the last two columns should be identical to each other.

Answer: Let p = "Pat finishes her marking" and q = "Pat does not feel guilty."

(a) So q is false, so p must also be false (the biconditional requires that p and q have the same values). So Pat did not finish her marking and the answer is "No."

(b) q is true so p is true. Yes.

(c) p is true so q is true. And since q is "not guilty", the answer is No.

(d) p is false so q is false. And feeling "not-not guilty" is just "guilty", so Yes.

2.8.2 Programming Applications

The if-then construction is very common in programming. In pseudocode, it usually takes the form

```
if x > 3 then print ''Hello World''
```

When you are debugging, it is tempting to think that this particular code fragment behaves more like the biconditional: if "Hello World" was output, was $x > 3$? It's tempting to think so, but what if "Hello World" was printed because of some other command? What then can we conclude about x?

Let's examine this in more detail. Recall that if $p \to q$ is true and q is true, we cannot conclude anything about p. Now consider the following piece of pseudocode.

```
x = 4
y = 0
if x > 3 then y = 5
print ''y = '', y
```

The output will be "$y = 5$".

But, won't the following pseudocode have the same output?

```
x = 2
y = 5
if x > 3 then y = 5
```

```
print ‘‘y = ‘’, y
```

It will. The conditional statement did not change the value of y, but the value was set to 5 initially, so the output is still be “$y = 5$”. Once again, knowing that q is true does not allow us to conclude anything about p from the conditional $p \to q$. However, if the output was “$y = 4$” or any other value not equal to 5, we can draw the conclusion that x was not greater than 3. If $p \to q$ is true and q is false, we know with certainty that p is false also.

The if-then-else construction behaves in a similar fashion. Consider the following code fragment:

```
if x > 3 then y = 5
else z = 7
print ‘‘y = ‘’, y, ‘‘z = ’’, z
```

Only if the output tells you that $y \neq 5$ or $z \neq 7$ will you know with certainty something about x.

For special cases, the if-then-else construction can yield more information. Consider the following.

```
if x > 3 then y = 5
else y = 7
print ‘‘y = ’’, y
```

Since this piece of pseudocode assigns different values (5 or 7) to the **same** variable y, finding out the resulting value of y will determine whether x was greater than 3. In this special case, the if-then behaves like the biconditional: if $y = 5$ then you know that $x > 3$, and if $y \neq 5$, then $x \leq 3$.

Exercises for Section 2.8

Write out the truth tables for the following logical expressions. (You might want to do it as just one or two really big tables.)

1. $p \to q$
2. $\sim p \to \sim q$
3. $\sim q \to \sim p$
4. $q \to p$
5. $\sim p \vee q$
6. $p \wedge \sim q$
7. $p \leftrightarrow q$
8. $\sim p \leftrightarrow \sim q$
9. $p \oplus q$
10. $p \vee \sim q$
11. $\sim p \oplus \sim q$
12. $(p \to q) \wedge (q \to p)$
13. $(p \to q) \vee (q \to p)$
14. $(p \to q) \wedge (\sim p \to \sim q)$
15. $(p \to q) \vee (\sim p \to \sim q)$
16. Looking at your results questions 1-15, which expressions are logically equivalent to $p \leftrightarrow q$?
17. Looking at your results for questions 1-15, which expressions are logically equivalent to $p \to q$?
18. Looking at your results for questions 1-15, which expressions are logically equivalent to $q \to p$?

Consider the following conditional statements. I hope you agree that they all make a certain amount of sense. However, if they were rewritten as **biconditional** statements, would they continue to make sense? Answer True or False.

19. If Barney is a dog, then he has four legs.

20. If Rich is asleep, then he is not playing ping-pong.
21. If Alycia gets 90% or better as her final mark, she will get an A+.
22. If Bossy is mooing, then she is a cow.
23. If Pat sleeps in, she is late for class.
24. If Frank does not pay his bill on time, he will be charged a late charge.
25. If Susan bought her computer less than a year ago, her warranty is still in effect.
26. If Raymond eats a burger for dinner, he will be too full for dessert.

In the following exercises, let p denote "Pat eats a burger for dinner" and let q denote "Pat is too full for dessert." Translate the following sentences into logical symbols.

27. If and only if Pat eats a burger for dinner, she will be too full for dessert.
28. Pat will not be too full for dessert if and only if she did not eat a burger for dinner.
29. If Pat eats a burger for dinner, then she will be too full for dessert.
30. If Pat is not too full for dessert, then she did not eat a burger for dinner.

Are the following two sentences biconditional statements? (In other words, could you replace them by an equivalent "if and only if" construction?)

31. If Frank does not pay his bill on time, then he will be charged a late charge, and if he does pay his bill on time, he will not be charged a late charge.
32. If Alycia gets 90% or better as her final mark, she will get an A+, and if she gets an A+, then she got 90% or better as her final mark.
33. The following conditional statement is true: If and only if Pat is eaten by bears, she will not finish her marking. Given that, answer the following questions.
    (a) Pat is eaten by bears. Did she finish her marking?
    (b) Pat is not eaten by bears. Did she finish her marking?
    (c) Pat finished her marking. Was she eaten by bears?

(d) Pat did not finish her marking. Was she eaten by bears?

34. The following conditional statement is true: If Rich is asleep, then he is not playing ping-pong and vice versa. Given that, answer the following questions.

    (a) Rich is playing ping-pong. Is he asleep?

    (b) Rich is asleep. Is he playing ping-pong?

    (c) Rich is not asleep. Is he playing ping-pong?

    (d) Rich is not playing ping-pong. Is he asleep?

35. The following conditional statement is true: Ettercaps are green if and only if toves are slithy. Given that, answer the following questions.

    (a) Toves are slithy. Are ettercaps green?

    (b) Toves are not slithy. Are ettercaps green?

    (c) Ettercaps are green. Are toves slithy?

    (d) Ettercaps are red. Are toves slithy?

36. If the statement "If and only if Superman has a cape, then he can fly" is a true statement, which of the following cannot occur? (You may choose more than one.)

    (a) Superman has a cape and he can fly.

    (b) Superman has a cape and he cannot fly.

    (c) Superman does not have a cape and cannot fly.

    (d) Superman does not have a cape and can fly.

Answers to Section 2.8 Exercises

Here are the truth tables for the expressions in questions 1 through 15.

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim p \to \sim q$	$\sim q \to \sim p$	$q \to p$	$\sim p \vee q$	$p \wedge \sim q$
0	0	1	1	1	1	1	1	1	0
0	1	1	0	1	0	1	0	1	0
1	0	0	1	0	1	0	1	0	1
1	1	0	0	1	1	1	1	1	0

p	q	$\sim p$	$\sim q$	$p \leftrightarrow q$	$\sim p \leftrightarrow \sim q$	$p \oplus q$	$p \vee \sim q$	$\sim p \oplus \sim q$
0	0	1	1	1	1	0	1	0
0	1	1	0	0	0	1	0	1
1	0	0	1	0	0	1	1	1
1	1	0	0	1	1	0	1	0

p	q	$(p \to q) \wedge (q \to p)$	$(p \to q) \vee (q \to p)$	$(p \to q) \wedge (\sim p \to \sim q)$	$(p \to q) \vee (\sim p \to \sim q)$
0	0	1	1	1	1
0	1	0	1	0	1
1	0	0	1	0	1
1	1	1	1	1	1

16. By comparing columns, the following expressions are logically equivalent to $p \leftrightarrow q$:
    (a) $\sim p \leftrightarrow \sim q$
    (b) $(p \to q) \wedge (q \to p)$
    (c) $(p \to q) \wedge (\sim p \to \sim q)$ (and you may or may not have noticed that it's also equal to $\sim p \oplus q$, which is kind of cool)

17. By comparing columns, the following expressions are logically equivalent to $p \to q$:
    (a) $\sim q \to \sim p$
    (b) $\sim p \vee q$

18. By comparing columns, the following expressions are logically equivalent to $q \to p$:

(a) $\sim p \rightarrow \sim q$

(b) $p \vee \sim q$

19. False
20. False
21. True
22. False
23. False
24. True
25. True
26. False
27. $p \leftrightarrow q$
28. $\sim q \leftrightarrow \sim p$
29. $p \rightarrow q$
30. $\sim q \rightarrow \sim p$
31. Yes
32. Yes
33. a) No b) Yes c) No d) Yes
34. a) No b) No c) Yes d) Yes
35. a) Yes b) No c) Yes d) No
36. b) and d)

Mixed Practice

1. Draw the gate diagram that corresponds to the Boolean expression $\overline{A} + \overline{B\,\overline{C}}$. Do not simplify!

2. Use a truth table to simplify the logical expression $(\sim p \wedge \sim q) \oplus (\sim p \wedge q)$.

3. Consider the statement, "This apple is red." Which of the following are logically equivalent to that statement? Circle any correct answers. You may choose more than one.

   (a) It is not true that this apple is not red.

   (b) This apple is red and this apple is red.

   (c) This apple is red or this apple is not red.

   (d) This apple is both red and shiny or this apple is red but not shiny.

   (e) This apple is red or this apple is both red and shiny.

   (f) This apple is red or this apple is not red but it is shiny.

4. Prove the following using the laws of logic. If you're stuck, try using a truth table for part marks.

$$A + (\overline{C} + 0)(\overline{B} + B) = \overline{A}\,\overline{C} + \overline{\overline{A} + \overline{A}}$$

5. The following statement is true: "If you eat at Joe's, then you will have a good meal." Given that, can the following situations occur? Answer "Yes" or "No".

   (a) You did not eat at Joe's and you had a good meal.

   (b) You did not eat at Joe's and you had a bad meal.

   (c) You ate at Joe's and you had a bad meal.

6. Consider the statement $p \to q$: "If you break a mirror, then you will have seven years of bad luck." Which of the following statements are logically equivalent to $p \to q$? Circle all of the correct answers.

   (a) If you don't break a mirror, you won't have seven years of bad luck.

   (b) If you do not have seven years of bad luck, then you did not break a mirror.

(c) If you have seven years of bad luck, then you broke a mirror.

(d) Either you did not break a mirror or you had seven years of bad luck or both.

7. Consider the statement: "If and only if a quantity is conserved, then a symmetry is exhibited." Answer the following questions with "Yes", "No", or "Maybe".

   (a) A quantity is not conserved. Is a symmetry exhibited?

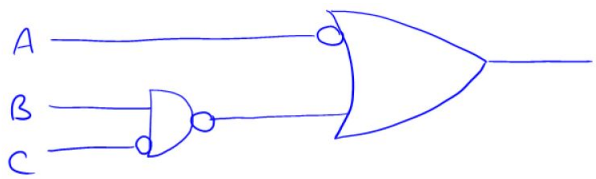
   (b) A symmetry is exhibited. Is a quantity conserved?

   (c) A symmetry is not exhibited. Is a quantity conserved?

8. Use a truth table to simplify the logical expression $(p \leftrightarrow \sim q) \wedge (p \leftrightarrow q)$.

Answers

1. Gate diagram for $\overline{A} + \overline{B\,\overline{C}}$:



2. Here's the truth table:

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$\sim p \wedge q$	$(\sim p \wedge \sim q) \oplus (\sim p \wedge q)$
0	0	1	1	1	0	1
0	1	1	0	0	1	1
1	0	0	1	0	0	0
1	1	0	0	0	0	0

The third and seventh columns are the same, so the expression simplifies to $\sim p$

3. (a), (b), (d), and (e) are correct.

Here is the reasoning. Let $p =$"This apple is red" and $q =$"This apple is shiny".

(a) The sentence translates to the expression $\sim(\sim p)$, which is p by complement.

(b) $p \wedge p$, which is p by idempotent.

(c) $p \vee \sim p$, which is 1 by identity and does not equal p

(d) $(p \wedge q) \vee (p \wedge \sim q) \Leftrightarrow p \wedge (q \vee \sim q)$ distributive

$\Leftrightarrow p \wedge 1$ complement

$\Leftrightarrow p$ identity

(e) $p \vee (p \wedge q) \Leftrightarrow p$ absorption

(f) $p \vee (\sim p \wedge q) \Leftrightarrow p \vee q$ absorption, which does not equal p

4. $A + (\overline{C} + 0)(\overline{B} + B) = \overline{A}\,\overline{C} + \overline{\overline{A} + \overline{A}}$ (this is the original statement)

$A + (\overline{C})(\overline{B} + B) = \overline{A}\,\overline{C} + \overline{\overline{A} + \overline{A}}$ identity

$A + (\overline{C})(1) = \overline{A}\ \overline{C} + \overline{\overline{A} + \overline{A}}$ complement

$A + \overline{C} = \overline{A}\ \overline{C} + \overline{\overline{A} + \overline{A}}$ identity

$A + \overline{C} = \overline{A}\ \overline{C} + \overline{\overline{A}}$ idempotent

$A + \overline{C} = \overline{A}\ \overline{C} + A$ complement

$A + \overline{C} = \overline{C} + A$ absorption (you can stop here if you wish)

$A + \overline{C} = A + \overline{C}$ commutative (you can skip this step)

Please note that many different solutions are possible!

5. If $p \to q$ is true, you cannot have p true and q false, so the answers are (a) Yes, (b) Yes, and (c) No.
6. (b) is the contrapositive, and (d) is the "or" form, so both (b) and (d) are correct.
7. If the biconditional $p \leftrightarrow q$ is true, then p and q are either both true or both false. So the answers are: (a) No, (b) Yes, (c) No.
8. Here's the truth table:

p	q	$\sim q$	$p \leftrightarrow \sim q$	$p \leftrightarrow q$	$(p \leftrightarrow \sim q) \wedge (p \leftrightarrow q)$
0	0	1	0	1	0
0	1	0	1	0	0
1	0	1	1	0	0
1	1	0	0	1	0

So the expression simplifies to 0.

Chapter 3

Sequences and Series

3.1 Introduction to Sequences and Series

3.1.1 Sequences

Let's start out with the definition of a sequence:

> **sequence**: an ordered list of numbers, often with a pattern

In a sequence, the number of terms can be finite or infinite. If a sequence is finite, then either the last term or the total number of terms must be specified so that it's clear where the sequence stops.

> **Example:** Which of the following sequences are infinite? Which are finite?
>
> (a) $7, 11, 15, 19, \ldots$
>
> (b) $1, 4, 9, 16, 25, 36, \ldots 100$
>
> (c) $4,\ 2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \ldots\ \frac{1}{256}$
>
> Answer: (b) and (c) are finite, because their last terms are given. (a), however, goes on forever so is infinite.

To begin with, let's examine some sequences in detail. We will begin by looking at sequences that **do** have a pattern.

> **Example:** What is the pattern for the following sequences? What is the next term for each sequence?

(a) $7, 11, 15, 19, \ldots$

(b) $1, 4, 9, 16, 25, 36, \ldots 100$

(c) $4,\ 2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \ldots\ \frac{1}{256}$

(d) $3, -6, 12, -24, \ldots$

(e) $3, -6, -15, -24, \ldots$

Answer:

(a) The pattern is that you add 4 to the previous term to get the next term. The next term is then 23.

(b) The pattern is that if you say that"1" is the first term and "4" is the second term, then n^2 will be the n^{th} term. So the next term after 36 is 49.

(c) The pattern is to divide each term by two (or multiply by one-half) to get the next term. So the term after 1/16 will be 1/32.

(d) The pattern is to multiply each term by -2 to get the next term. The next term is then 48.

(e) The pattern is to subtract 9 from the previous term, so the next one is -33.

Note that in this previous example, the last two sequences looked very similar for three of their first four terms. However, the third term is different so the pattern for the two sequences is not the same and subsequent terms could look very different.

3.1.2 Notation for Sequences

For each term in a sequence, we will use the notation of a lower-case a followed by a subscript which is called the index. So, depending on what we want our starting index to be, our sequence can be written as

$$a_0, a_1, a_2 \ldots, a_n$$

or

$$a_1, a_2, a_3, \ldots, a_n$$

or even

$$a_5, a_6, a_7, \ldots, a_n$$

In this textbook, we will use the convention that the starting index is m, so our sequences can be written as

$$a_m, a_{m+1}, a_{m+2}, \ldots, a_n$$

Because we are examining sequences from a computing perspective, we should be aware that computing languages don't use a single convention: many start counting at $m = 0$, while others start at $m = 1$.[1] In this textbook we will simply specify the start value of our index for each sequence instead of using any one convention.

3.1.3 Counting the Terms in a Sequence

Since it's possible to start the index for a sequence at any value, we need to be careful when determining k, the total number of terms in a sequence. The rule is:

$$\#\text{terms} = \text{last} - \text{first} + 1$$

and since we are using the convention that m is the first index and n is the final index (or, alternatively, the index of interest), then

$$k = n - m + 1$$

Starting with Index of One

Let us consider a sequence that starts with an index of one:

$$a_1, a_2, a_3, \ldots, a_n$$

This convention has the advantage that if you label each term as follows:

$$\overbrace{a_1}^{\text{first}}, \overbrace{a_2}^{\text{second}}, \overbrace{a_3}^{\text{third}}, \overbrace{a_4}^{\text{fourth}}, \overbrace{a_5}^{\text{fifth}}, \ldots, \overbrace{a_n}^{\text{final}}$$

you can see that the term a_5 has an index $n = 5$ and is also the fifth term, so the number of the term (fifth) and the index (5) are consistent with each other. This makes it more difficult to make a counting error. Also, the total number of terms in the sequence $a_1, a_2, a_3, \ldots, a_n$ is given by $k = n - 1 + 1$, so $k = n$ and is consistent with what we would expect.[2]

[1] Examples of languages that have a starting index of zero are Python and the C family (C, C++, C#). Languages which start their index values at one include Fortran, Smalltalk, and Lua. There are also languages such as Algol which start at a user-defined value.

[2] In mathematics, it is most common to start counting with a_1 being the first term. Programming languages primarily designed for mathematics, such as Matlab, usually start with an index of one.

Starting with Index of Zero

However, let us now consider sequences that start with zero:

$$a_0, a_1, a_2, a_3, \ldots, a_n$$

Numbering the terms, we find that

$$\overbrace{a_0}^{\text{first}}, \overbrace{a_1}^{\text{second}}, \overbrace{a_2}^{\text{third}}, \overbrace{a_3}^{\text{fourth}}, \overbrace{a_4}^{\text{fifth}}, \ldots, \overbrace{a_n}^{\text{final}}$$

and a_5 is no longer the fifth term. In fact, a_5 is the **sixth** term, which is why it is common in programming to separate the "count" of a term (first, second, third, etc.) from the index value (0 for a_0, etc.).

Also, the total number of terms in $a_0, a_1, a_2, a_3, \ldots, a_n$ is given by

$$\begin{aligned} k &= n - m + 1 \\ &= n - 0 + 1 \\ &= n + 1 \end{aligned}$$

so k, the "count" of the term is no longer equal to n, the index of the final term. So be warned: if you are not careful with this convention, you are likely to make a type of mistake which programmers commonly call an "off-by-one" error.

Starting with Index of Two or More

As we have seen, a_5 is only the fifth term in sequences that start with an index of one. If the sequence starts at some other value, then a_5 could even be the first or second term. This does lead to a small problem in that the term a_n is commonly called the the n^{th} term in a sequence, which is only true for a starting index of one.[3]

3.1.4 Defining a Sequence

There are three ways to define a sequence:

1. List all of the terms, or enough terms to set up the pattern. If the sequence is finite, then either the final term or the total number of terms must be given.

[3] This leads to the awkward convention of calling a_0 the **zeroth** term.

2. Give a general formula for the n^{th} term.

3. Give a recursive formula for the n^{th} term.

We have already looked at sequences defined using the first method in the examples given earlier. Let's now examine the two types of formula, general and recursive.

3.1.5 General Formula

A general formula is a formula that gives a_n as a function of n only. What this means is that the only variable on the right-hand-side of the general formula is the variable n, and all other values in the equations are constants.

Let's look at the following examples to examine some sequences defined in this way.

Example: Give the first four terms of the sequence given by the general formula $a_n = 4n + 7$ for $n \geq 0$.

Answer:

$$
\begin{aligned}
a_n &= 4n + 7, \text{so} \\
a_0 &= 4 \times 0 + 7 = 7 \\
a_1 &= 4 \times 1 + 7 = 11 \\
a_2 &= 4 \times 2 + 7 = 15 \\
a_3 &= 4 \times 3 + 7 = 19
\end{aligned}
$$

The first four terms are then 7, 11, 15, and 19. This is the same sequence that was given as part (a) in the first example of this section.

Example: Give all terms of the sequence given by the formula $a_n = \left(\frac{1}{3}\right)^n$ for $1 \leq n \leq 5$.

Answer: This is a finite sequence, since restrictions have been placed on the values of n. The terms are then:

$$a_1 = \left(\frac{1}{3}\right)^1 = \frac{1}{3}$$
$$a_2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$
$$a_3 = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$$
$$a_4 = \left(\frac{1}{3}\right)^4 = \frac{1}{81}$$
$$a_5 = \left(\frac{1}{3}\right)^5 = \frac{1}{243}$$

You can see from the previous examples that the general formula allows you to calculate a_n for any value of n. The very useful thing about the general formula is that you don't need to know the previous term to calculate a particular term. For instance, if you want to know a_{50} for the sequence $7, 11, 15, 19, \ldots$, you can determine that the pattern is to add 4 to the previous term to get the next term. However, to get a_{50}, you'd have to calculatea_{49} first, but a_{49} requires a_{48}, and so on. But if you instead use the formula $a_n = 4n + 7$ for $n \geq 0$, which gives the same sequence, then a_{50} is just

$$a_n = 4n + 7$$
$$a_{50} = 4 \cdot 50 + 7 = 207$$

and there's no need to calculate preceding terms. Handy![4]

3.1.6 Recursive Definition

A recursive formula gives a formula for the next term in terms of the previous one. For example, in our old friend $7, 11, 15, 19, \ldots$, the next term is found by adding 4 to the previous term: $a_n = a_{n-1} + 4$. However, that's not enough information to uniquely define the series because you don't know where to start. A complete definition must include the first term also. Therefore, the

[4] It's important to note, however, that a_{50} is **not** the fiftieth term. Because we are starting our index from zero, a_{50} is the **fifty-first** term since $k = n-m+1 = 50-0+1 = 51$.

recursive definition for our old friend $7, 11, 15, 19, \ldots$ would be

$$\begin{cases} a_0 = 7 \\ a_n = a_{n-1} + 4 \quad \text{for } n \geq 1 \end{cases}$$

Recursive definitions, then, must specify the first term (or terms, when necessary) **and also** the rule which allows you to calculate the next term from the previous term or terms.

Example: Calculate the first four terms of the sequence given by

$$\begin{cases} a_0 = 3 \\ a_n = (a_{n-1} - 1)^2 + 10 \quad \text{for } n \geq 1 \end{cases}$$

Answer: The first term is already given, $a_0 = 3$. Then

$$\begin{aligned} a_1 &= (3-1)^2 + 10 = 2^2 + 10 = 14 \\ a_2 &= (14-1)^2 + 10 = 13^2 + 10 = 179 \\ a_3 &= (179-1)^2 + 10 = 178^2 + 10 = 31694 \end{aligned}$$

So the first four terms are 3, 14, 179, 31694.

Example: Give a recursive formula for the sequence $2, 6, 18, 54, \ldots$

Answer: The pattern is that the next term equals the previous term times three. We can start our index at either 0 or 1, so let's choose 1. Therefore,

$$\begin{cases} a_1 = 2 \\ a_n = 3a_{n-1} \quad \text{for } n \geq 2 \end{cases}$$

Recursive definitions have the same drawback that we've seen before: if we want to know the 200^{th} term, we need to calculate the 199^{th} first, and so on. Only the general formula allows us to calculate each term directly without knowing the previous one.

3.1.7 Fibonacci sequence

The Fibonacci sequence is the most famous example of a recursive sequence: $1, 1, 2, 3, 5, 8, 13, \ldots$

The pattern can be quite difficult to spot – you get the next term from the **sum** of the two previous terms. The recursive formula for this sequence is therefore

$$\begin{cases} a_1 = 1 \\ a_2 = 1 \\ a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 3 \end{cases}$$

Here, the first **two** terms must be given to start off with so that you are then able to calculate the third term from the previous two.

3.1.8 Series

A series is the sum of the terms of a sequence, which can be finite or infinite. Here are two examples:

(a) $16 + 20 + 24 + 28 + \ldots + 64$

(b) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ...$

You can see that the first example is a finite series, while the second one is infinite.

3.1.9 Notation for Series

The sum of the first k terms of a sequence is denoted by S_k (also sometimes called the k^{th} partial sum). If the series is finite, it could be the sum of **all** of the terms. S_∞ is how we write the sum of an infinite series, like the second example above.

Example: For the series $16 + 20 + 24 + 28 + \ldots + 64$, calculate S_3 and S_4.

Answer:

$$S_3 = 16 + 20 + 24 = 60$$
$$S_4 = 16 + 20 + 24 + 28 = 88$$

However, it's easy to see that this method becomes very cumbersome for large values of k. We'll develop some more efficient methods for particular types of series in the next two sections.

3.1.10 Sigma Notation

It's easy to take a sequence in list form and transform it into a series by changing all of the commas to plus signs. However, what if you are given the general formula instead? For example, let's take $7, 11, 15, 19, \ldots$ which we know to be $a_n = 4n + 3$ for $n \geq 1$. Since the general form is so useful for finding a_n when n is large, it would be nice if we could retain that information while writing our sum.

To do so, we'll introduce a new notation called "sigma notation'. It uses the Greek letter sigma (the uppercase one): Σ, which is commonly used to mean "sum of'.

Let's look at an example of sigma notation and discuss what all of the parts mean. Consider the following sum:

$$\sum_{i=1}^{5} (4i + 3)$$

The letter i is an index here, and it runs from the value given at the bottom of the sigma to the number at the top of the sigma in steps of 1. Here, i runs from 1 to 5. We are summing, then, the value of $4i + 3$ for each value of i as it runs from 1 to 5:

$$\begin{aligned}\sum_{i=1}^{5} (4i + 3) &= \overbrace{(4 \times 1 + 3)}^{i=1} + \overbrace{(4 \times 2 + 3)}^{i=2} + \overbrace{(4 \times 3 + 3)}^{i=3} + \overbrace{(4 \times 4 + 3)}^{i=4} + \overbrace{(4 \times 5 + 3)}^{i=5} \\ &= 7 + 11 + 15 + 19 + 23 \\ &= 75\end{aligned}$$

Let's look at more examples.

Example: Calculate $\sum_{i=0}^{2} (2i - 5)$.

Answer:

$$\begin{aligned}\sum_{i=0}^{2}(2i-5) &= \overbrace{(2\times 0-5)}^{i=0}+\overbrace{(2\times 1-5)}^{i=1}+\overbrace{(2\times 2-5)}^{i=2}\\ &= -5+(-3)+(-1)\\ &= -9\end{aligned}$$

Example: Calculate $\sum_{j=6}^{9}(8-j)^2$.

Answer:

$$\begin{aligned}\sum_{j=6}^{9}(8-j)^2 &= \overbrace{(8-6)^2}^{j=6}+\overbrace{(8-7)^2}^{j=7}+\overbrace{(8-8)^2}^{j=8}+\overbrace{(8-9)^2}^{j=9}\\ &= 4+1+0+1\\ &= 6\end{aligned}$$

Example: Calculate $\sum_{k=12}^{16} 3$.

Answer:

$$\begin{aligned}\sum_{j=12}^{16} 3 &= \overbrace{3}^{k=12}+\overbrace{3}^{k=13}+\overbrace{3}^{k=14}+\overbrace{3}^{k=15}+\overbrace{3}^{k=16}\\ &= 15\end{aligned}$$

The tricky thing about the last one is deciding how many terms there are. Recall that you can either write out all of the possible values of the index, or use the useful rule:

$$k = n - m + 1$$

and as the last example had the index running from 12 to 16, then the number of terms k is

$$k = 16 - 12 + 1$$

$$k = 5$$

Example: Write the following series in sigma notation:

$$4 + 9 + 16 + 25 + ...100$$

Answer: Let's pick our index first. If we want to be lazy, instead of starting our index at 0 or 1, we could start at 2 and our series would be

$$\sum_{k=2}^{10} k^2$$

Other acceptable answers would involve changing our starting point for the index to give

$$\sum_{j=1}^{9} (j+1)^2$$

or

$$\sum_{i=0}^{8} (i+2)^2$$

or even

$$\sum_{l=157}^{165} (l-155)^2$$

if 157 happens to be your favourite number.

Example: Write the following sequence in sigma notation:

$$\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots$$

Answer:

$$\sum_{j=3}^{\infty} \frac{1}{j}$$

To write an infinite series in sigma notation, you just replace the final value of the index with ∞.

Exercises for Section 3.1

Predict the next three terms of the following sequences.

1. $18, 16, 14, \ldots$
2. $1, 4, 9, 16, \ldots$
3. $12, 24, 48, 96, \ldots$
4. $144, 36, 9, \ldots$
5. $1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}, ...$
6. $5, -10, 20, \ldots$
7. $13, 25, 37, 49, \ldots$
8. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...$

Give a formula for the general term (the n^{th} term a_n in terms of n) of the following sequences. Use $n = 1$ as your starting index.

9. $1, 4, 9, 16, \ldots$
10. $1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}, ...$
11. $2, 4, 6, 8, \ldots$
12. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...$

Find the first four terms of the following recursively defined sequences.

13. $\begin{cases} a_1 = 2 \\ a_n = a_{n-1} + 5 \quad \text{for } n \geq 2 \end{cases}$

14. $\begin{cases} a_1 = 10 \\ a_n = 3a_{n-1} \quad \text{for } n \geq 2 \end{cases}$

15. $\begin{cases} a_1 = 2 \\ a_2 = 3 \\ a_n = a_{n-1} \times a_{n-2} \quad \text{for } n \geq 3 \end{cases}$

16. $\begin{cases} a_1 = 2 \\ a_n = \frac{1}{a_{n-1}} + 1 \quad \text{for } n \geq 2 \end{cases}$

In each of the following, the general formula for the n^{th} term of a sequence is given. Find the first four terms.

17. $a_n = 3n - 5$ for $n \geq 1$
18. $a_n = 3^{n-2}$ for $n \geq 1$
19. $a_n = n!$ for $n \geq 1$
20. $a_n = \frac{1}{n^2}$ for $n \geq 1$

In each of the following, the general formula for the nth term of a sequence is given. Calculate the specified terms.

21. Find a_7 for the sequence $a_n = 5\left(2^{n+1}\right)$ for $n \geq 1$
22. Find a_{100} for the sequence $a_n = 4n + 15$ for $n \geq 1$
23. Find a_{2500} for the sequence $a_n = \frac{n+2}{n+1}$ for $n \geq 1$
24. Find a_{10} for the sequence $a_n = 2n^3$ for $n \geq 1$

Calculate S_3 and S_6 for the following series.

25. $3 + 6 + 9 + \ldots$
26. $1 + 4 + 9 + 16 + \ldots$
27. $5 - 10 + 20 - 40 + \ldots$
28. $5 + 3 + 1 + \ldots$

Write out each sum in full and then evaluate.

29. $\sum_{n=3}^{7} n$
30. $\sum_{j=4}^{10} (-1)^j$
31. $\sum_{i=0}^{4} 2^i$
32. $\sum_{k=20}^{25} (3k - 10)$

Write each series in sigma notation. (Answers may vary.)

33. $1 + 8 + 27 + 64 + \ldots 1000$

34. $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ...$

35. $2 + 4 + 6 + 8 + \ldots$

36. $2 + 4 + 6 + 8$

Evil alert! The following questions are just for those wanting a challenge. This type of question will not be tested.

37. (nasty) Write the sequence $1, 4, 9, 16, \ldots$ using a **recursive** definition.

38. (thorny) Write the sequence $1, 2, 6, 24, \ldots$ using a **general** formula.

39. (tricksy) Consider the following sequence:

$$4, 5, 20, 100, 2000$$

   (a) What's the next term in this sequence?

   (b) What's the recursive formula for this sequence?

Answers to Section 3.1 Exercises

1. $12, 10, 8$ (pattern is to subtract 2)
2. $25, 36, 49$ (n^{th} term is equal to n^2
3. $192, 384, 768$ (multiply by 2)
4. $\frac{9}{4}, \frac{9}{16}, \frac{9}{64}$ (divide by 4)
5. $\sqrt{7}, 2\sqrt{2}, 3$ (n^{th} term is $\sqrt{n}$)
6. $-40, 80, -160$ (multiply by -2)
7. $61, 73, 85$ (add 12)
8. $\frac{1}{6}, \frac{1}{7}, \frac{1}{8}$
9. $a_n = n^2$
10. $a_n = \sqrt{n}$
11. $a_n = 2n$
12. $a_n = \frac{1}{n+1}$
13. $2, 7, 12, 17$
14. $10, 30, 90, 270$
15. $2, 3, 6, 18$
16. $2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}$
17. $-2, 1, 4, 7$
18. $\frac{1}{3}, 1, 3, 9$
19. $1, 2, 6, 24$
20. $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$
21. $a_7 = 1280$
22. $a_{100} = 415$
23. $a_{2500} = \frac{2502}{2501}$
24. $a_{10} = 2000$
25. $S_3 = 18, S_6 = 63$

26. $S_3 = 14, S_6 = 91$

27. $S_3 = 15, S_6 = -105$

28. $S_3 = 9, S_6 = 0$

29. $\sum_{n=3}^{7} n = 3 + 4 + 5 + 6 + 7 = 25$

30. $\sum_{j=4}^{10} (-1)^j = 1 + (-1) + 1 + (-1) + 1 + (-1) + 1 = 1$

31. $\sum_{i=0}^{4} 2^i = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31$

32. $\sum_{k=20}^{25} 3k - 10 = 50 + 53 + 56 + 59 + 62 + 65 = 345$

33. $\sum_{i=1}^{10} i^3$

34. $\sum_{j=2}^{\infty} \frac{1}{j}$

35. $\sum_{k=1}^{\infty} 2k$

36. $\sum_{k=1}^{4} 2k$

37. You could either do

$$\begin{cases} a_1 = 1 \\ a_n = \left(\sqrt{a_{n-1}} + 1\right)^2 & \text{for } n \geq 2 \end{cases}$$

or another possibility is

$$\begin{cases} a_1 = 1 \\ a_n = a_{n-1} + 2n - 1 & \text{for } n \geq 2 \end{cases}$$

38. $a_n = n!$ for $n \geq 1$

39. The next term is 200,000.

$$\begin{cases} a_1 = 4 \\ a_2 = 5 \\ a_n = a_{n-1} \times a_{n-2} \end{cases}$$

3.2 Arithmetic Sequences and Series

3.2.1 Arithmetic Sequences

Let's start out with a definition:

> **arithmetic sequence**: a sequence in which the next term is found by adding a constant (the common difference d) to the previous term

Here are some examples of arithmetic sequences:

(a) $7, 11, 15, 19, \ldots$

(b) $11, 4, -3, -10, \ldots -59$

(c) $12, 12.3, 12.6, 12.9, \ldots$

The first one has a common difference of 4, the second -7, and the third 0.3. Note that in each of them, we can find the common difference d by taking **any** term and subtracting the previous term from it.

Example: For the following sequences, state whether each of them is arithmetic.

(a) $-3, -10, -17, -24, \ldots$

(b) $4, 5, 7, 10, \ldots$

(c) $2, 4, 8, 16, \ldots$

(d) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \frac{1}{20}$

Answer:

(a) Yes, because the common difference d is -7.

(b) No, because you're not adding the same number each time.

(c) No, because you're multiplying by 2 to get the next term, not adding.

(d) No, because the difference between each pair of terms is different.

Again, you can define an arithmetic sequence in one of three ways: by listing the terms, by giving a recursive definition, or by giving a general definition.

3.2.2 Recursive Definitions for Arithmetic Sequences

Let's look first at an example.

Example: Give a recursive definition for the sequence $2, 10, 18, 26, \ldots$

Answer: Recall that a recursive definition has two parts: listing the first term and giving the pattern. In this case, the pattern is adding $d = 8$ to the previous term to get the next term. We can start our index anywhere, so let's choose zero for this example. The recursive definition is therefore

$$\begin{cases} a_0 = 2 \\ a_n = a_{n-1} + 8 \quad \text{for } n \geq 1 \end{cases}$$

To generalize, the recursive formula for **any** arithmetic sequence is

$$\begin{cases} a_m = < \text{insert value here} > \\ a_n = a_{n-1} + d \quad \text{for } n \geq m + 1 \end{cases}$$

3.2.3 General Formulae for Arithmetic Sequences

Let's examine the previous example in more detail to see if we can recognize any patterns and come up with a general formula. Rewriting each term, we get

$$\begin{array}{ccccc} 2, & 10, & 18, & 26, & \ldots \\ 2 & 2+8, & 2+8\times 2, & 2+8\times 3, & \ldots \end{array}$$

So the 3^{rd} term equals the first plus 8 times 2, the 4^{th} term equals the first plus 8 times 3, and the n^{th} term will equal the first plus 8 times $(n-1)$. In other words,

$$\begin{array}{cccccc} 2, & 10, & 18, & 26, & \ldots & a_n \\ 2 & 2+8, & 2+8\times 2, & 2+8\times 3, & \ldots & 2+8\times(n-1) \end{array}$$

and so we find for this particular sequence, $a_n = 2 + 8 \times (n-1)$, which simplifies to $a_n = 8n - 6$.

We can generalize this formula: the n^{th} term will equal the first plus d times $(n-m)$, so

$$a_n = a_m + (n-m)\,d \quad \text{where } n \geq m$$

for any **arithmetic sequence**.

Example: Write a general formula for the sequence $3, 8, 13, 18, \ldots$

Answer: This sequence is arithmetic with the first term 3 and common difference 5. Let's use a starting index of zero.

$$\begin{aligned} a_n &= a_m + (n-m)\,d \\ &= a_0 + (n-0)\,d \\ &= 3 + (n)\,5 \\ &= 3 + 5n \\ &= 5n + 3 \end{aligned}$$

The general formula is then $a_n = 5n + 3$ for $n \geq 0$.

Example: What is the 50^{th} term in the sequence in the sequence $3, 8, 13, 18, \ldots$?

Answer: This is the same sequence from the previous example. We may then use the formula we derived, $a_n = 5n + 3$ for $n \geq 0$. But we do have to be careful about our index n. Recalling that the number of terms k is given by

$$k = n - m + 1$$

where we used $m = 0$ as our starting index, then

$$50 = n - 0 + 1$$

$$n = 49$$

and so the 50^{th} term will be a_{49}.

$$\begin{aligned} a_n &= 5n + 3 \\ a_{49} &= 5 \times 49 + 3 \\ &= 245 + 3 \\ &= 248 \end{aligned}$$

The 50^{th} term is 248.

Example: What is the common difference in the arithmetic sequence in which the first term is 18 and the twelfth term is -59?

Answer: The easiest way to count these terms correctly is to have a starting index of one, and then the first term is a_1 and the twelfth term is a_{12}.

$$\begin{aligned} a_n &= a_m + (n-m)\,d \\ a_{12} &= a_1 + (12-1)\,d \\ -59 &= 18 + (12-1)\,d \\ -77 &= 11d \\ d &= -7 \end{aligned}$$

The common difference is -7.

Example: Which term has a value of 404 in the sequence $-37, -28, -19, \ldots$?

Answer: Let's use a starting index of one. So a_1 is -37 and d is $+9$. Then we want to find the value of n for which a_n equals 404.

$$\begin{aligned} a_n &= a_m + (n-m)\,d \\ &= a_1 + (n-1)\,d \\ 404 &= -37 + (n-1)\,9 \\ 441 &= 9\,(n-1) \\ 49 &= n-1 \\ n &= 50 \end{aligned}$$

The **fiftieth** term is 404.

3.2.4 Arithmetic Series

Recall that S_k is the sum of the first k terms of a series. Let's look at a couple of examples of arithmetic series to see if we can identify any patterns.

Suppose we wish to take some partial sums of the series $2+10+18+26+\ldots$. Let's first calculate S_6. We could just find the first six terms and add them up, but notice the following:

$$S_6 = 2 + 10 + 18 + 26 + 34 + 42$$

The sum of the first and last numbers is 44. The sum of the second and second-to-last is also 44. So is the sum of the third and third-last. So when you take the terms in pairs, each pair has the same sum, $(a_m + a_n)$, and there are $k/2$ pairs in total. Then

$$S_k = \frac{k}{2}(a_m + a_n)$$

What if, however, there are an odd number of terms? Let's also calculate S_7:

$$S_7 = 2 + 10 + 18 + 26 + 34 + 42 + 50$$

The sum of the first and last is 52, as is the sum of the each "inner pair'. Notice that the middle, unpaired value, is $\frac{1}{2}$ of 52. So in a sense, the middle term is $\frac{1}{2}$ of a pair, for a total of $3\frac{1}{2}$ pairs. But that's just 7/2, which is our $k/2$ in the original formula! So we're still good. The relationship

$$S_k = \frac{k}{2}(a_m + a_n)$$

still works, for both odd and even values of k.

Generalizing, we find that

$$S_k = \frac{k}{2}(a_m + a_n)$$

where k can be even or odd and

$$k = n - m + 1 \quad \text{for } n \geq m$$

> **Example:** Find the sum of the first forty terms of the series $2 + 10 + 18 + 26 + \ldots$.
>
> Answer: This is just the same sequence as before, with first term $a_m = 2$ and common difference $d = 8$. In order to use our previous formula, however, we need to calculate the last term a_n. If we start with an index of one, then the fortieth term will be a_{40}, and we will need that value to calculate S_{40}.
>
> $$\begin{aligned} a_n &= a_m + (n - m)\,d \\ &= a_1 + (n - 1)\,d \\ a_{40} &= 2 + 39 \times 8 \\ &= 314 \end{aligned}$$

So,

$$
\begin{aligned}
S_k &= \frac{k}{2}\left(a_m + a_n\right) \\
S_{40} &= \frac{40}{2}\left(2 + 314\right) \\
&= 20 \times 316 \\
&= 6320
\end{aligned}
$$

The sum of the first forty terms is 6320. (Much easier than writing out the first forty terms and adding them up!)

In the previous example, we used the formula for a_n to calculate the last term and put its value into the formula for S_k. We could do that in a more general way:

$$
\begin{aligned}
S_k &= \frac{k}{2}\left(a_m + a_n\right) \\
&= \frac{k}{2}\left[a_m + \left(a_m + \left(n - m\right) d\right)\right] \\
&= \frac{k}{2}\left[2a_m + \left(n - m\right) d\right]
\end{aligned}
$$

and the last expression, which gives S_k as a function of the first term, the number of terms, and the common difference, can also be used to evaluate series.

Example: Find the sum of the first one hundred terms of the sequence $5, -6, -17, -28, \ldots$.

Answer: This sum will just be $5 + -6 + -17 + -28 + \ldots$, with $a_m = 5$, $d = -11$, and $k = 100$. If we start our index at one, then

$$
\begin{aligned}
k &= n - m + 1 \\
100 &= n - 1 + 1 \\
n &= 100
\end{aligned}
$$

and we can substitute this into the equation for S_k:

$$
\begin{aligned}
S_k &= \frac{k}{2}\left[2a_m + \left(n - m\right) d\right] \\
S_{100} &= \frac{100}{2}\left[2 \times 5 + 99 \times (-11)\right] \\
&= -53950
\end{aligned}
$$

Example: Calculate $\sum_{j=3}^{18} 5n + 10$.

Answer: The first term will be for $j = 3$ and will equal $5(3)+10 = 25$. Next is $j = 4$ and will equal $5(4) + 10 = 30$, $j = 5$ equaling $5(5) = 35$, and so on. The last term will be for $j = 18$ and will equal $5(18) + 10 = 100$.

In other words, our series is $25+30+35+\ldots 100$. Is it arithmetic? Yes, with common difference $d = 5$.

What else do we need for our calculation? The number of terms is

$$k = n - m + 1$$
$$k = 18 - 3 + 1$$
$$k = 16$$

Then

$$S_n = \frac{n}{2}\left(a_m + a_n\right)$$
$$S_{16} = \frac{16}{2}\left(25 + 100\right) = 1000$$

Example: Pat the math instructor asks her students to do five word problems the first week, six the second week, seven the third week, and so on, increasing the number of word problems each week by one.

(a) How many word problems will diligent students be doing in the last week of classes (the n^{th}11 week)?

(b) How many word problems will diligent students have completed during the course of the term (11 weeks)?

Answer:

(a) The number of word problems is a sequence: $5, 6, 7, \ldots$. In fact, it's an arithmetic sequence with $a_m = 5$ and $d = 1$. If we start our counting from one, then in the eleventh week,

$$a_n = a_m + (n - m)\, d$$
$$a_n = a_1 + (n - 1)\, d$$
$$a_{11} = 5 + 10 \times 1$$
$$= 15$$

Diligent students will solve 15 word problems in the last week of classes.

(b) The **total** number of word problems solved is

$$\begin{aligned} S_k &= \frac{k}{2}\left(a_m + a_n\right) \\ S_{11} &= \frac{11}{2}\left(5 + 15\right) \\ &= 110 \end{aligned}$$

Diligent students will have solved 110 word problems in total.

3.2.5 Summary

For an **arithmetic sequence**, the nth term is given by

$$a_n = a_m + (n - m)\, d \quad \text{for } n \geq m$$

For an **arithmetic series**, the sum of the first k terms (k^{th} partial sum) is

$$S_k = \frac{k}{2}\left(a_m + a_n\right)$$

or

$$S_k = \frac{k}{2}\left[2a_m + (n - m)\, d\right]$$

where $k = n - m + 1$ and $n \geq m$.

Exercises for Section 3.2

State whether the following sequences are arithmetic or not. If they are, state the first term and common difference.

1. $8, 9, 11, 13, 16, \ldots$
2. $-3, -10, -17, -24, \ldots$
3. $3, 6, 12, 24, \ldots$
4. $1, 2, 6, 24, \ldots$
5. $81, 72, 63, 54, \ldots$
6. $1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2, \frac{9}{4}, ...$

Give both the general formula and the recursive formula for the n^{th} term a_n of the following arithmetic sequences. Assume that the first term of the sequence is a_1. For the general formula, be sure to simplify your answer.

7. $1, 3, 5, 7, \ldots$
8. $5, -6, -17, -28, \ldots$
9. $-40, -37, -34, -31, \ldots$
10. $24, 28, 32, 36, \ldots$

For the following arithmetic sequences, calculate a_{50} and a_{261}, assuming that the first term is a_1.

11. $18, 16, 14, 12, \ldots$
12. $12, 12.3, 12.6, 12.9, \ldots$

State whether the following recursively defined sequences are arithmetic or not. (Is there an easy way to tell?)

13. $\begin{cases} a_0 = 5 \\ a_n = a_{n-1} + 4 \quad \text{for } n \geq 1 \end{cases}$
14. $\begin{cases} a_1 = 12 \\ a_n = 2a_{n-1} \quad \text{for } n \geq 2 \end{cases}$
15. $\begin{cases} a_1 = 75 \\ a_n = a_{n-1} - 20 \quad \text{for } n \geq 2 \end{cases}$

16. $\begin{cases} a_0 = 6 \\ a_n = a_{n-1} + 1 \quad \text{for } n \geq 1 \end{cases}$

17. $\begin{cases} a_0 = 7 \\ a_n = 2 - a_{n-1} \quad \text{for } n \geq 1 \end{cases}$

18. $\begin{cases} a_1 = 3 \\ a_n = (a_{n-1})^2 \quad \text{for } n \geq 2 \end{cases}$

19. For the following sequence, calculate the 201^{st} term: $5, 15, 25, 35, \ldots$

20. For the following sequence, which term equals 137? $1, 9, 17, 25, \ldots$

21. What is the common difference for the arithmetic sequence with $a_1 = 200$ and $a_{12} = -240$?

22. Calculate the first term for the arithmetic sequence with common difference 7 whose sixteenth term is 102.

23. Calculate the first four terms of the arithmetic sequence in which the sixth term is 17 and the sixtieth term is 179.

24. Calculate the first four terms of the arithmetic sequence in which the one hundredth term is 403 and the sixty-fourth term is 259.

25. Give a general formula for the arithmetic sequence in which the twentieth term is -107 and the thirty-fifth term is -152.

26. Give a recursive formula for the arithmetic sequence in which the eleventh term is 44 and the fifty-second term is 290.

27. Calculate S_{20} for the series $100 + 97 + 94 + \ldots$

28. Evaluate the series $12 + 17 + 22 + \ldots 82$.

29. Evaluate the series $144 + 138 + 132 + \ldots 78$.

30. Calculate S_{100} for the series $-20 - 16 - 12 - \ldots$

31. Calculate the sum of the odd numbers between 100 and 500.

32. Find the sum of the integers from 50 to 125, inclusive.

Calculate the following sums.

33. $\sum_{k=0}^{53} (5k - 1)$

34. $\sum_{j=10}^{92} 6j$

35. $\sum_{i=30}^{140} (2i + 7)$

36. $\sum_{k=3}^{502} (17 - 3k)$

37. In a supermarket display, there are 37 cans in the bottom layer, 35 in the next layer up, 33 in the next, and so on. How many layers are there if there are 7 cans in the top row?

38. In the previous problem, how many cans are there altogether?

39. In an old-fashioned theatre, there are 25 seats in the first row, 26 in the next, 27 in the one after, and so on. If there are 20 rows in total, how many seats are there altogether?

Answers to Section 3.2 Exercises

1. not arithmetic
2. yes, $d = -7$
3. no
4. no
5. yes, $d = -9$
6. yes, $d = \frac{1}{4}$
7. $a_n = 2n - 1$ and $\begin{cases} a_1 = 1 \\ a_n = a_{n-1} + 2 \end{cases}$
8. $a_n = 16 - 11n$ and $\begin{cases} a_1 = 5 \\ a_n = a_{n-1} - 11 \end{cases}$
9. $a_n = 3n - 43$ and $\begin{cases} a_1 = -40 \\ a_n = a_{n-1} + 3 \end{cases}$
10. $a_n = 4n + 20$ and $\begin{cases} a_1 = 24 \\ a_n = a_{n-1} + 4 \end{cases}$
11. $a_n = 20 - 2n$, so $a_{50} = -80$ and $a_{261} = -502$
12. $a_n = 11.7 + 0.3n$, so $a_{50} = 26.7$ and $a_{261} = 90$
13. first four terms are $5, 9, 13, 17$, so arithmetic with $d = 4$
14. first four terms are $12, 24, 48, 96$, so not arithmetic
15. first four terms are $75, 55, 35, 15$, so arithmetic with $d = -20$
16. first four terms are $6, 7, 8, 9$, so arithmetic with $d = 1$
17. first four terms are $7, -5, 7, -5$, so not arithmetic
18. first four terms are $3, 9, 81, 6561$, so not arithmetic
19. $a_n = 10n - 5$, so $a_{201} = 2005$
20. $a_n = 8n - 7$, so $n = 18$
21. $d = -40$

22. $a_1 = -3$
23. $a_1 = 2$ and $d = 3$, so the first four terms are $2, 5, 8, 11$
24. $a_1 = 7$ and $d = 4$, so the first four terms are $7, 11, 15, 19$
25. $a_n = -3n - 47$
26. $\begin{cases} a_1 = -16 \\ a_n = a_{n-1} + 6 \end{cases}$
27. $S_{20} = 1430$
28. $S_{15} = 705$
29. $S_{12} = 1332$
30. $S_{100} = 17800$
31. $S_{200} = 60000$
32. $S_{76} = 6650$
33. $S_{53} = 7101$
34. $S_{83} = 25398$
35. $S_{111} = 19647$
36. $S_{500} = -370,250$
37. $n = 16$
38. $S_{16} = 352$
39. $S_{20} = 690$

3.3 Geometric Sequences and Series

3.3.1 Geometric Sequences

Let's start out with a definition:

> **geometric sequence**: a sequence in which the next term is found by multiplying the previous term by a constant (the common ratio r)

Here are some examples of geometric sequences:

(a) $9, 18, 36, 72, \ldots$

(b) $12, 18, 27, \frac{81}{2}, \ldots$

(c) $10, -30, 90, -270, \ldots, -196830$

(d) $-3, -12, -48, -192, \ldots$

(e) $48, -36, 27, \ldots$

The common ratios of each of these sequences, in order from a) to e), is 2, $\frac{3}{2}$, -3, 4, $-\frac{3}{4}$, respectively. Note that in each of them, we can find the common ratio r by taking **any** term and dividing it by the previous term.

Like any other sequences, geometric sequences can be finite or infinite. Example c) above is finite, as the last term is specified. The others are infinite sequences.

> **Example:** For each of the following sequences, state whether it is arithmetic, geometric, or neither.
>
> (a) $45, 15, 5, \ldots$
>
> (b) $5, 3, 1, -1, \ldots$
>
> (c) $1, 8, 27, 64, \ldots, 1000$
>
> (d) $-1, 1, -1, 1, -1, 1, \ldots$
>
> Answer:
>
> (a) Geometric, because the common ratio r is $\frac{1}{3}$.
>
> (b) Arithmetic, because the common difference d is -2.

(c) Neither, because there isn't either a common difference or ratio between terms. (In fact, the pattern is that $a_n = n^3$ for $n \geq 1$.)

(d) Geometric, because the common ratio r is -1.

Again, you can define a geometric sequence in one of three ways: by listing the terms, by giving a recursive definition, or by giving a general definition.

3.3.2 Recursive Definitions for Geometric Sequences

Let's look at an example.

Example: Give a recursive definition for the sequence $2, 10, 50, 250, \ldots$

Answer: Recall that a recursive definition has two parts: listing the first term and giving the pattern. In this case, the pattern is multiplying the previous term by $r = 5$ to get the next term. Let's use 0 as our starting index. The recursive definition is therefore

$$\begin{cases} a_0 = 2 \\ a_n = 5a_{n-1} \quad \text{for } n \geq 1 \end{cases}$$

Generally, the recursive definition for **any** geometric sequence is

$$\begin{cases} a_m = < \text{insert value here} > \\ a_n = a_{n-1} \times r \quad \text{for } n \geq m+1 \end{cases}$$

3.3.3 General Formulae for Geometric Sequences

Let's examine the previous example in more detail to see if we can recognize any patterns and come up with a general formula. Rewriting each term, we get

$$\begin{array}{ccccc} 2, & 10, & 50, & 250, & \ldots \\ 2, & 2 \times 5, & 2 \times 5^2, & 2 \times 5^3, & \ldots \end{array}$$

So the 3^{rd} term equals the first times 5 squared, the 4^{th} term equals the first times 5 cubed, and the n^{th} term will equal the first times 5 raised to the $(n-1)$ power. In general, for sequences with first term a_m, the nth term equals the first term times r raised to the $(n-m)$ power, namely

$$a_n = a_m r^{n-m}$$

for all **geometric** sequences.

Example: Write a general formula for the sequence $3, 6, 12, \ldots$

Answer: This sequence is geometric with the first term 3 and common ratio 2. If we choose $n = 1$ for our first term, then

$$\begin{aligned} a_n &= a_m r^{n-m} \\ &= a_1 r^{n-1} \\ &= 3 \times (2)^{n-1} \end{aligned}$$

The general formula is then $a_n = 3 \times 2^{n-1}$ for $n \geq 1$.

Example: What is the 20^{th} term in the sequence in the sequence $3, 6, 12, \ldots$?

Answer: This is the same sequence from the previous example. We may then use the formula we derived above with $n = 20$.

$$\begin{aligned} a_n &= a_m r^{n-m} \\ &= a_1 r^{n-1} \\ a_{20} &= 3 \times 2^{20-1} \\ a_{20} &= 3 \times 2^{19} \\ a_{20} &= 1,572,864 \end{aligned}$$

The 20^{th} term is 1,572,864, which provides a nice example for how fast geometric sequences can grow, even for small values of r.

Example: Write a general formula for the sequence $8, 12, 18, 27, \ldots$ What is the fifteenth term in this sequence? The fiftieth?

Answer: If we start our counting at $n = 1$, then the fifteenth term is a_{15} and the fiftieth term is a_{50}.

$$\begin{aligned} a_n &= a_m r^{n-m} \\ &= a_1 r^{n-1} \\ a_n &= 8\left(\frac{3}{2}\right)^{n-1} \\ a_{15} &= 8\left(\frac{3}{2}\right)^{14} \approx 2335.43 \\ a_{50} &= 8\left(\frac{3}{2}\right)^{49} \approx 3.40065 \times 10^9 \end{aligned}$$

So the general formula is $a_n = 8\left(\frac{3}{2}\right)^{n-1}$ for $n \geq 1$ and the fifteenth and fiftieth terms are approximately 2335.43 and 3.4×10^9, respectively.

3.3.4 Geometric Series

Recall that S_k is the sum of the first k terms of a series. Let's look at how a formula for S_k is derived, using a series that starts with $n = 1$.

$$\begin{array}{lllllllll} S_k = & a_1 & + a_2 & + a_3 & + a_4 & + \ \ldots & + a_{n-2} & + a_{n-1} & + a_k \\ S_k = & a_1 & + a_1 r & + a_1 r^2 & + a_1 r^3 & + \ \ldots & + a_1 r^{k-3} & + a_1 r^{k-2} & + a_1 r^{k-1} \end{array}$$

Let's take that last expression for S_k and multiply it by $-r$ to get

$$-rS_k = -a_1 r - a_1 r^2 - a_1 r^3 - a_1 r^4 - ... - a_1 r^{k-2} - a_1 r^{k-1} - a_1 r^k$$

Then if we add the rows for S_k and $-rS_k$, we get

$$\begin{array}{ccccccccccc} S_k & = & a_1 & +a_1 r & +a_1 r^2 & +a_1 r^3 & +\ldots & +a_1 r^{k-3} & +a_1 r^{k-2} & +a_1 r^{k-1} & \\ -rS_k & = & & -a_1 r & -a_1 r^2 & -a_1 r^3 & -a_1 r^4 & -\ldots & -a_1 r^{k-2} & -a_1 r^{k-1} & -a_1 r^k \\ \hline S_k - rS_k & = & a_1 & & & & & & & & -a_1 r^k \end{array}$$

since all of the terms in between these two (a_1 and $a_1 r^k$) will cancel. Then

$$S_k - rS_k = a_1 \left(1 - r^k\right)$$

$$S_k \left(1 - r\right) = a_1 \left(1 - r^k\right)$$

and

$$S_k = \frac{a_1 \left(1 - r^k\right)}{(1 - r)}$$

To generalize, the formula for the sum of the first n terms for **any geometric series** that starts with first term a_m is

$$S_k = \frac{a_m \left(1 - r^k\right)}{(1 - r)}$$

Example: Find the sum of the first 20 terms of the series $3 + 6 + 12 + \ldots$

Answer: This is a geometric series with $a_m = 3$ and $r = 2$. We want to find S_{20}.

$$S_k = \frac{a_m\left(1 - r^k\right)}{(1 - r)}$$
$$S_{20} = \frac{3\left(1 - 2^{20}\right)}{(1 - 2)} = 3,145,725$$

The sum of the first 20 terms is 3,145,725.

Example: Find the sum of the first forty terms of the series $8 - 12 + 18 - 27 \ldots$.

Answer: This is a geometric series with $a_m = 8$ and $r = -\frac{3}{2}$. We want to find S_{40}.

$$S_k = \frac{a_m\left(1 - r^k\right)}{(1 - r)}$$
$$S_{20} = \frac{8\left(1 - (-1.5)^{40}\right)}{(1 - (-1.5))} = -3.53835 \times 10^7$$

The sum of the first forty terms is -3.54×10^7.

3.3.5 Sum of an Infinite Geometric Series

Let's take a look at the infinite series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...$ What happens when we try to evaluate this sum using the S_k formula? We can put $a_m = \frac{1}{2}$, $r = \frac{1}{2}$, and $n = \infty$ into the formula, but we will run into a roadblock when we try to evaluate $(\frac{1}{2})^\infty$.

Let's take a closer look at the behaviour of $(\frac{1}{2})^n$ for large values of n. As n gets larger, the fraction $\left(\frac{1}{2}\right)^n = \frac{1}{2^n}$ gets ever smaller. In fact, as n approaches ∞, $(\frac{1}{2})^n$ will approach zero.

This is true for any r provided that $|r| < 1$. (If you're not familiar with the absolute value bars, an equivalent expression is that $-1 < r < 1$.)

Recalling that

$$S_k = \frac{a_m\left(1 - r^k\right)}{(1 - r)}$$

and letting the r^n term go to zero, then

$$S_\infty = \frac{a_m}{1 - r} \text{ for} - 1 < r < 1$$

for any **infinite geometric series** with a_m the first term, provided that r meets the restriction above.

Let's now revisit the series that started this discussion and evaluate it in the following example.

Example: Evaluate $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...$

Answer: This series is geometric with $a_m=\frac{1}{2}$ and $r=\frac{1}{2}$. Then

$$S_\infty=\frac{a_m}{1-r}=\frac{1/2}{1-1/2}=\frac{1/2}{1/2}=1$$

The sum of this series is 1.

Example: Evaluate $24+16+\frac{32}{3}+\dots$

Answer: This series is geometric with $a_m=24$ and $r=\frac{2}{3}$.

$$S_\infty=\frac{a_m}{1-r}=\frac{24}{1-\frac{2}{3}}=\frac{24}{\frac{1}{3}}=24\times\frac{3}{1}=72$$

Example: Evaluate $24-16+\frac{32}{3}+\dots$

Answer: This series is identical to the previous one except that r is now negative: $a_m=24$ and $r=-\frac{2}{3}$.

$$S_\infty=\frac{a_m}{1-r}=\frac{24}{1-\left(-\frac{2}{3}\right)}=\frac{24}{1+\frac{2}{3}}=\frac{24}{\frac{5}{3}}=24\times\frac{3}{5}=\frac{72}{5}=14.4$$

Example: Evaluate $12+18+27+\dots$

Answer: This series is geometric with $a_m=12$ and $r=\frac{3}{2}$. You may already realize what's going on, but in case you don't, let's naively put the values into the formula and see what we get:

$$S_\infty=\frac{a_m}{1-r}=\frac{12}{1-\frac{3}{2}}=\frac{12}{-\frac{1}{2}}=12\times-\frac{2}{1}=-24$$

Wait! How can the sum of a bunch of positive number be negative? The answer is that our restriction for r is that it must be between -1 and 1, but $r=1.5$. Because r does not satisfy the restriction, we cannot use the above formula for S_∞. Indeed, if you add up a bunch of positive numbers that are increasing as you go up, you can see that the sum just keeps getting bigger as we add more terms. You could then either say that the sum is infinite (dicey) or "does not exist" (safer).

But why is it safer to say "does not exist" in the last example? Let's look at three sums:

(a) $12 + 18 + 27 + \ldots$

(b) $-12 - 18 - 27 - \ldots$

(c) $12 - 18 + 27 + \ldots$

Each term in (a) is getting more positive, so the sum of that sequence will be $+\infty$. Each term in (b) is getting more and more negative, so the sum of that sequence will be $-\infty$. But in the last term, the sum oscillates back and forth: $S_1 = 12$, $S_2 = -6$, $S_3 = 21$, $S_4 = -19.5$, and so on. The sign of S_k is either positive or negative depending on whether the number of terms you've added is even or odd. Rather than debating whether infinity is odd or even (?!), we will just say that the sum "does not exist".

Example: Evaluate $\sum_{j=0}^{\infty} 27\left(\frac{1}{3}\right)^j$.

Answer: Ick! The best place to start is to figure out the first few terms to determine the pattern:

$$\begin{aligned}
&\text{when } j = 0,\ 27\left(\tfrac{1}{3}\right)^0 = 27 \times 1 = 27 \\
&\text{when } j = 1,\ 27\left(\tfrac{1}{3}\right)^1 = 27 \times \tfrac{1}{3} = 9 \\
&\text{when } j = 2,\ 27\left(\tfrac{1}{3}\right)^2 = 27 \times \tfrac{1}{3^2} = 3
\end{aligned}$$

so our sequence is $27, 9, 3, \ldots$ This is geometric with $a_m = 27$ and $r = \frac{1}{3}$. Then

$$S_\infty = \frac{a_m}{1-r} = \frac{27}{1-\frac{1}{3}} = \frac{27}{\frac{2}{3}} = 27 \times \frac{3}{2} = \frac{81}{2} = 40.5$$

Example: Evaluate $\sum_{k=5}^{\infty} \frac{1}{2}k$.

Answer: Once again, let's figure out the first few terms to determine the pattern:

$$\begin{aligned}
&\text{when } k = 5,\ \tfrac{1}{2}k = \tfrac{1}{2}5 = 2.5 \\
&\text{when } k = 6,\ \tfrac{1}{2}k = \tfrac{1}{2}6 = 3 \\
&\text{when } k = 7,\ \tfrac{1}{2}k = \tfrac{1}{2}7 = 3.5
\end{aligned}$$

so our sequence is $2.5, 3, 3.5, \ldots$. Wait! This is arithmetic! Not only that, but the numbers are increasing. So the sum will be infinite, or if you prefer, the sum "does not exist".

3.3.6 Repeating Decimals

Let's examine $0.\overline{7}$ in some detail to see what we find:

$$\begin{aligned} 0.\overline{7} &= 0.777777777... \\ &= 0.7 + 0.07 + 0.007 + 0.0007 + \ldots \end{aligned}$$

But this is just the sum of an infinite series with $a_m = 0.7$ and $r = 0.1$. Rewriting a_1 and r in fraction form (you'll see why in a minute) gives $a_m = \frac{7}{10}$ and $r = \frac{1}{10}$. Then

$$S_\infty = \frac{a_m}{1-r} = \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{\frac{7}{10}}{\frac{9}{10}} = \frac{7}{10} \times \frac{10}{9} = \frac{7}{9}$$

So $0.\overline{7} = 7/9$. Interesting!

Example: Find an exact fraction for $0.\overline{6}$.

Answer:

$$\begin{aligned} 0.\overline{6} &= 0.66666666... \\ &= 0.6 + 0.06 + 0.006 + 0.0006 + ... \end{aligned}$$

But this is just the sum of an infinite series with $a_m = \frac{6}{10}$ and $r = \frac{1}{10}$. Then

$$S_\infty = \frac{a_m}{1-r} = \frac{\frac{6}{10}}{1-\frac{1}{10}} = \frac{\frac{6}{10}}{\frac{9}{10}} = \frac{6}{10} \times \frac{10}{9} = \frac{6}{9} = \frac{2}{3}$$

So $0.\overline{6} = 2/3$.

Example: Find an exact fraction for $0.\overline{18}$.

Answer:

$$\begin{aligned} 0.\overline{18} &= 0.1818181818... \\ &= 0.18 + 0.0018 + 0.000018 + ... \end{aligned}$$

But this is just the sum of an infinite series with $a_m = \frac{18}{100}$ and $r = \frac{1}{100}$. Then

$$S_\infty = \frac{a_m}{1-r} = \frac{\frac{18}{100}}{1-\frac{1}{100}} = \frac{\frac{18}{100}}{\frac{99}{100}} = \frac{18}{100} \times \frac{100}{99} = \frac{18}{99} = \frac{2}{11}$$

So $0.\overline{18} = 2/11$.

3.3.7 Summary

For a **geometric** sequence, the n^{th} term is given by

$$a_n = a_m r^{n-m}$$

for $n \geq m$

For a **geometric** series, the sum of the first k terms (k^{th} partial sum) is

$$S_k = \frac{a_m\left(1-r^k\right)}{(1-r)}$$

where a_m is the first term

For an **infinite geometric** series, the sum is

$$S_\infty = \frac{a_m}{1-r}$$

provided that $-1 < r < 1$ and a_m is the first term.

Exercises for Section 3.3

State whether the following sequences are geometric or not. If they are, state the first term and common ratio.

1. $8, 9, 11, 13, 16, \ldots$
2. $-3, -10, -17, -24, \ldots$
3. $3, 6, 12, 24, \ldots$
4. $1, 2, 6, 24, \ldots$
5. $81, 72, 63, 54, \ldots$
6. $72, 48, 32, ...$

Give both the general formula and the recursive formula for the n^{th} term a_n of the following sequences. Use the convention $n \geq 1$.

7. $1, 3, 9, 27, \ldots$
8. $64, 16, 4, 1, \ldots$
9. $3, -6, 12, -24, \ldots$
10. $24, 2.4, 0.24, \ldots$

For the following sequences, calculate a_{50} and a_{261}, assuming that the first term is a_1.

11. $12, 18, 27, \ldots$
12. $12, 8, \frac{16}{3}, \ldots$

State whether the following recursively defined sequences are geometric or not. (Is there an easy way to tell?)

13. $\begin{cases} a_1 = 5 \\ a_n = a_{n-1} + 4 \quad \text{for } n \geq 2 \end{cases}$

14. $\begin{cases} a_0 = 12 \\ a_n = 2a_{n-1} \quad \text{for } n \geq 1 \end{cases}$

15. $\begin{cases} a_0 = 75 \\ a_n = 10a_{n-1} \quad \text{for } n \geq 1 \end{cases}$

16. $\begin{cases} a_1 = 7 \\ a_n = 2 - a_{n-1} \quad \text{for } n \geq 2 \end{cases}$

17. $\begin{cases} a_1 = 8 \\ a_n = -a_{n-1} \quad \text{for } n \geq 2 \end{cases}$

18. $\begin{cases} a_0 = 3 \\ a_n = (a_{n-1})^2 \quad \text{for } n \geq 1 \end{cases}$

19. For the following sequence, calculate the 201^{st} term: $5, 15, 45, \ldots$

20. For the following sequence, calculate the 20^{th} term: $7, -14, 28, \ldots$

21. Calculate S_{20} for the series $100 + 50 + 25 + \ldots$

22. Calculate S_{20} for the series $100 + 200 + 400 + \ldots$

Calculate the sum, if it exists, for the following series.

23. $-6 + 4 - \frac{8}{3} + ...$

24. $100 + 50 + 25 + \ldots$

25. $100 + 200 + 400 + \ldots$

26. $12 + 3 + \frac{3}{4} + \ldots$

Calculate the following sums, if they exist.

27. $\sum_{k=0}^{10} 2^{k+2}$

28. $\sum_{j=1}^{\infty} 15\left(\frac{3}{5}\right)^j$

29. $\sum_{i=2}^{\infty} 25(0.1)^i$

30. $\sum_{i=0}^{\infty} 4(-3)^i$

31. If the number of vampires in Transylvania doubles every month, then how many vampires will be in Transylvania in 3 years, starting from one individual? Comment on your result if the total population of Transylvania is 2 million people.

32. As I was going to St. Ives, I met a man with seven wives. Each wife had seven sacks. Each sack had seven cats. Each cat had seven kits. Kits, cats, sacks, wives: does this form a geometric sequence?

33. The paper used in the photocopier by Pat's office is said to be 0.097 mm thick. If it is folded over repeatedly, doubling its thickness each time, how thick will the paper be if it's folded 7 times? Bonus: why, then, were the Mythbusters having so many problems trying to fold the paper this many times?

Answers to Section 3.3 Exercises

1. no
2. no
3. yes, $r = 2$
4. no
5. no
6. yes, $r = \frac{2}{3}$
7. $a_n = (3)^{n-1}$ and $\begin{cases} a_1 = 1 \\ a_n = 3a_{n-1} \end{cases}$
8. $a_n = 64\left(\frac{1}{4}\right)^{n-1}$ and $\begin{cases} a_1 = 64 \\ a_n = \frac{a_{n-1}}{4} \end{cases}$
9. $a_n = 3(-2)^{n-1}$ and $\begin{cases} a_1 = 3 \\ a_n = -2a_{n-1} \end{cases}$
10. $a_n = 24(0.1)^{n-1}$ and $\begin{cases} a_1 = 24 \\ a_n = 0.1 \times a_{n-1} \end{cases}$
11. $a_n = 12\left(\frac{3}{2}\right)^{n-1}$, so $a_{50} \approx 5.1 \times 10^9$ and $a_{261} \approx 7.3 \times 10^{46}$
12. $a_n = 12\left(\frac{2}{3}\right)^{n-1}$, so $a_{50} \approx 2.8 \times 10^{-8}$ and $a_{261} \approx 1.97 \times 10^{-45}$
13. no
14. yes, with $r = 2$
15. yes, with $r = 10$
16. no
17. yes, with $r = -1$
18. no
19. $a_n = 5(3)^{n-1}$, so $a_{201} = 5(3)^{200} = 1.33 \times 10^{96}$
20. $a_n = 7(-2)^{n-1}$, so $a_{20} = 7(-2)^{19} = -3,670,016$

21. $S_{20} = 200$ (the exact answer is $\frac{26214375}{131072}$ or 1.99980926513671875, but if you round to three decimals, the answer is 200.000)

22. $S_{20} = 104,857,500$

23. $S_\infty = \frac{a_1}{1-r} = \frac{-6}{1-(-2/3)} = -\frac{18}{5} = -3.6$

24. $S_\infty = 200$

25. S_∞ does not exist $(r > 1)$

26. $S_\infty = 16$

27. $S_{11} = 2^2 + 2^3 + 2^4 + ... + 2^{12} = \frac{a_1(1-r^n)}{1-r} = \frac{2^2(1-2^{11})}{1-2} = 8188$

28. $S_\infty = 22.5$

29. $S_\infty = \frac{5}{18} = 0.2\overline{7}$

30. S_∞ does not exist $(r < -1)$

31. 3 years is 36 months, so we have a 36-term sequence starting with $1, 2, 4, 8, \ldots$ The n^{th} term will be $a_n = 1(2)^{n-1}$, so the 36^{th} term will be $a_{36} = 1(2)^{35} = 34,359,738,368$, which is a tad larger than the total population of Transylvania.

32. 1 man

    7 wives

    #sacks = #wives × #sacks/wife = 7 × 7 = 49

    #cats = #sacks × #cats/sack = 49 × 7 = 343

    #kits = #cats × #kits/cat = 343 × 7 = 2401

    So kits, cats, sacks, and wives is the sequence $2401, 373, 49, 7$, which is geometric with four terms: $a_1 = 2401$ and $r = \frac{1}{7}$.

33. The paper is initially 0.097 mm thick with no folds. After one fold, the thickness will be 0.097×2, after two folds $0.097 \times 2 \times 2$, etc. So our starting term (zero folds) will be $a_0 = 0.097$ and then will double with r = 2 thereafter, where n is not only the index but also the number of folds made. So $a_n = 0.097(2)^n$, and the term with seven folds will be $a_7 = 0.097(2)^7 = 12.416$, so we can conclude that the paper thickness will be 12.4 mm, or just over 1 cm thick. (The Mythbusters realized that the problems with paperfolding lie with the fold itself, and making

the fold lie as flat as possible. If I remember correctly, they resorted to C-clamps and hitting the fold with a hammer to flatten it.)

Mixed Practice

1. Label the following sequences as "arithmetic", "geometric" or "neither".
   (a) $1, 1, 2, 3, 5, 8, \ldots$
   (b) $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \ldots$
   (c) $58, 48, 38, \ldots$
2. Consider the sequence given by the following.
   $$a_n = 30 - 3n, \qquad 1 \leq n \leq 3$$
   (a) Is this formula recursive or general?
   (b) Calculate all terms of this sequence.
3. Evaluate the following sum, if it exists. If it doesn't exist, state why not. Show your work!
   $$\sum_{i=2}^{\infty} 8(-3)^i$$
4. Calculate the first three terms of the following sequence.
   $$\begin{cases} a_1 = 3 \\ a_n = (a_{n-1})^2 \quad \text{for} \quad n \geq 2 \end{cases}$$
5. Write a recursive formula for the sequence defined below.
   $$a_n = 7 * 3^n \quad \text{for} \quad n \geq 1$$
6. State whether the following are arithmetic, geometric, or neither. Also, give a formula for the nth term of the sequence. Use a starting index of one.
   (a) $15, 9, 3, -3, \ldots$
   (b) $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$
   (c) $48, 12, 3, \frac{3}{4}, \ldots$

7. Calculate the following sums, if possible. If not possible, state why not. Show your work.

   (a) $\sum_{j=0}^{4}(3j)$

   (b) $2+4+6+\ldots+88$

   (c) $\sum_{m=0}^{\infty} 300(0.99^m)$

   (d) $\frac{1}{25}-\frac{1}{20}+\frac{1}{16}-\frac{5}{64}+\ldots$

8. Calculate the sum of the odd numbers between 1000 and 5000. Show your work.

Answers

1. (a) neither

   (b) geometric

   (c) arithmetic

2. (a) general

   (b) 27, 24, 21

3. undefined, because it is geometric with $r = -3$, and $|r| < 1$ is false

4. 3, 9, 81

5. $\begin{cases} a_1 = 21 \\ a_n = 3a_{n-1} \quad \text{for} \quad n \geq 2 \end{cases}$

6. (a) arithmetic
   either $a_n = 21 - 6n$ (general) or
   $\begin{cases} a_1 = 15 \\ a_n = a_{n-1} - 6 \quad \text{for} \quad n \geq 2 \end{cases}$ (recursive)

   (b) neither
   $a_n = \frac{n-1}{n}$

   (c) geometric,
   either $a_n = 48\left(\frac{1}{4}\right)^{n-1}$ (general) or
   $\begin{cases} a_1 = 48 \\ a_n = \frac{1}{4}a_{n-1} \quad \text{for} \quad n \geq 2 \end{cases}$ (recursive)

7. (a) $\sum_{j=0}^{4}(3j) = 0 + 3 + 6 + 9 + 12 = 30$
   (b) arithmetic with $d = 2$ and $n = 44$, so $S_{44} = 1980$
   (c) geometric with $a_m = 300$ and $r = 0.99$, so $S_\infty = 30000$
   (d) geometric with $r = -\frac{5}{4}$, so sum does not exist

8. $1001 + 1003 + 1005 + \ldots + 4999$
   arithmetic series with $d = 2$
   number of terms:
   $a_n = a_m + (n - m)d$
   let's start with $m = 1$
   $4999 = 1001 + (n - 1)2$
   solving for n gives $n = 2000$
   then $S_n = \frac{n}{2}(a_m + a_n)$
   and $S_{2000} = 6\ 000\ 000$

Chapter 4

Big O Notation and Algorithmic Complexity

4.1 Big O and Rates of Growth

4.1.1 Basic Concepts

Before we get to definitions, let's start off with a conceptual example. Suppose you are moving and you want to rent a truck for a day. You have looked up the rental rates and your options are:

- A flat rate of \$80 per day.
- A rate of \$2 per kilometre.

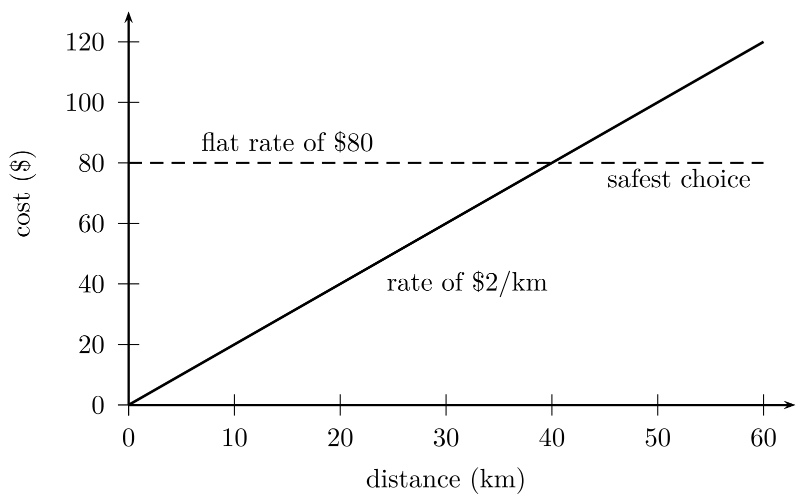
Which rate is better?

Hopefully it's clear that the best rate must depend on how many kilometres you will be driving. If you know that you will only be making one trip with a total distance of 10 km, then you would only have to pay \$20. However, if you will be driving for more than 40 km, then the flat rate of \$80 per day would cost less than the rate per kilometre.

The big idea is that if, when you are first renting the truck, you are unsure about how far you will be driving, then the safest bet is to go for the flat rate. If you end up driving a short distance, then you could possibly have saved money on the other rate, but if you end up driving a long distance, then you could possibly pay a lot more than \$80.

The worst-case scenario, then, would be that you ended up driving for a much longer distance than you had planned. But if you had taken that into account when choosing the rate, you could choose the rate that would be cheaper for the worst-case scenario.

Let's take a look at the graph for this situation.



The point on the graph where the two lines intersect is the breakeven point, where the two options cost the same amount of money. On the graph, we can see that this occurs when the distance is 40 km and the cost is $80. And as we've discussed, the safest choice under most circumstances is the flat rate of $80. To find this safest choice, we look to the far right of the graph, for large values of the x-coordinate. Whichever is the lowest curve in that region of the graph will be the safest choice under the worst case (in our example, it's the scenario in which you end up driving more than you anticipated).

4.1.2 Definition of Big O

In computing, rather than looking at minimzing cost we could use this same idea to look at the number of steps or operations that a computer program takes while running: the more steps required, the longer the program will take to execute. In basic language,

> **Big O** describes the number of steps required to complete an algorithm for a task of size n when n gets large

When studying Big O, we should also keep in mind that there may be other considerations than having the shortest run time. If we are using a large data set, we may instead be trying to minimize the amount of working memory that our program requires. Other associated notations are Big Omega (Ω), which essentially gives the best-case scenario, and Big Theta (Θ), which is used when Big O and Big Omega are the same.

Why is it that in computing we are concerned with the worst-case scenario? One reason is that Big O problems show up as bad performance in software. For example, if you have designed a site where people can enter maintenance histories for their cars, then you want to be prepared for the case that a rental company uses your software to enter the histories for 11,000 cars. Users can frequently come up with usage patterns that you didn't anticipate and it is wise to be prepared for that possibility.

4.1.3 Choosing the Most Efficient Procedure

Let us suppose that you need to put a list of items into alphabetical order, and you are trying to decide on the best way to do this. You could, for example, alphabetize the list by hand. You could also write a program to sort the list for you. Which method should you choose?

- if the list is very short (5 items? 10?) **and** you only ever have to do this task once, it is probably fastest to do it by hand
- if the list is long or you will need to do this many times in future, then it may be worth the time to write the computer program

If you know in advance what the size of the list will be, then you should be able to choose between these two methods with confidence. Where Big O comes in is the situation in which you know you have to do this task, but you are not sure what the size of the list will be.

Notice also that in this example, we were trying to minimize the amount of time spent alphabetizing the list. You might make a different choice, for example, if your goal was to improve your programming skills. You might then choose to write a program even if the list is very short and you only had to do the task once.

> **Example:** You are at home and you have a flash drive containing data that you need to give to your classmate on campus. You have two choices: upload the files to DropBox at 10 MBps (10 megabytes per second, which works out to 600 MB

per minute) or walk 20 minutes to campus and hand the flash drive to your classmate.

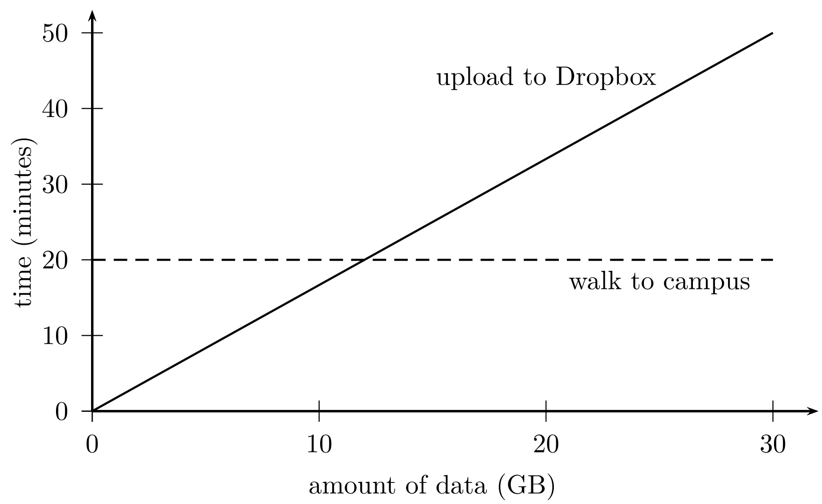
Use the graph below to answer the following questions

(a) What is the breakeven point, where the two methods will take the same amount of time? (Just give a rough estimate of the amount of data.)

(b) What is the most efficient method if you only have 5 GB worth of data?

(c) What is the most efficient method if you have 25 GB worth of data?

(d) Which method should you choose if you don't remember how much data you have but your classmate needs to know **right now** which method you are going to choose?



Answer:

(a) From the graph, it's just over 10 GB (answer is exactly 12 GB or around 12 GB, depending on whether you use the definition of 1 GB = 1000 MB or 1 GB = 1024 MB).

(b) Uploading is faster, because at 5 GB, the upload line is below the walk line.

(c) Walking to campus is faster, because at 25 GB, the walk line is below the upload line.

(d) Walking is best, because you know it will take a maximum of 20 minutes no matter how much data there is. Otherwise your classmate might be stuck for hours waiting for the download to finish.

In the previous example, if you uploaded the data to DropBox, doubling the amount of data will double the time required. We could write this as an equation in which the time t is a multiple of the amount of data n. In Big O notation, we drop the coefficient we are multiplying n by and write this as

$$O(n)$$

and say that this is "of order n".

In contrast, walking the flashdrive to campus takes a constant amount of time, so is a multiple of the number one. Big O notation then gives this as

$$O(1)$$

and we say that this is "of order one". We have also determined that for sufficiently large n, the graph of $O(n)$ will always be above the graph of $O(1)$, so $O(1)$ is the best choice in the worst-case scenario.

Other Big O relationships are possible, such as $O(n^2)$ and $O(2^n)$. We'll examine these in the next section.

Exercises for Section 4.1

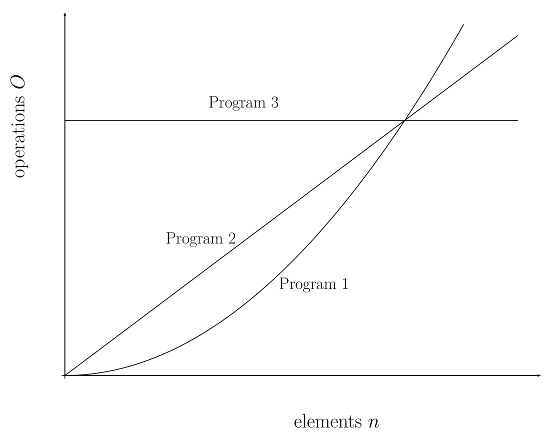
1. Many parts of the world have highways that are toll roads, so that you have to pay a fee to drive on them. For example, before 2008 the Coquihalla Highway in southern BC had a toll for cars of \$10 but saved drivers at least an hour in travel time over the alternate route.

   If you were driving in that part of BC, which route should you take (Coquihalla vs. alternate route) if you are

   (a) broke?

   (b) running late and are not broke?

2. This graph shows the number of operations O required to complete a task of size n elements for Programs 1, 2, and 3, where Program 1 is the curved line, Program 2 is the straight line through the origin, and Program 3 is the horizontal line.



   Indicate whether the following statements are true or false.

   (a) There is a certain size of task n where all three programs require the same number of steps.

   (b) Program 2 is a good choice for all sizes of n because it is the "middle ground" between Programs 1 and 3.

   (c) There is no value of n for which Program 2 is clearly the best choice.

(d) For large n, Program 3 will finish faster than the other two programs because the line for Program 3 is below the lines for the other programs on the right-hand side of the graph.

(e) Whether Program 1 is more efficient than Program 3 depends on the size of n.

3. You are living in an apartment block with a single washer and dryer in the basement. You have the choice of doing your laundry one load at a time using the machines downstairs, or you can drive to the laundromat and use many machines at once. Each load of laundry takes one hour to wash and dry using either your apartment's machines or the laundromat's. The laundromat is 30 minutes away by car.

   (a) Under what conditions is your apartment's washer/dryer the fastest way to do your laundry?

   (b) Under what conditions will the two different options take about the same amount of time?

   (c) If you have many loads of clothing, which is the better option?

4. You are playing a computer game and you have a choice of playing a fighter, a cleric, or a wizard. In combat, the fighter always does 50 points worth of damage no matter what level the fighter is. The cleric does 10 points of damage per level, while the damage the wizard does is equal to the square of the level.

   (a) Which character choice (fighter/cleric/wizard) does the most damage at low levels? At high levels?

   (b) At what level is the breakeven point between fighter and cleric?

   (c) At what level does the wizard start to do more damage than the fighter?

Answers to Section 4.1 Exercises

1. (a) alternate route

   (b) Coquihalla

2. (a) true

   (b) false

   (c) true

   (d) true

   (e) true

3. (a) You only have one load.

   (b) You have two loads.

   (c) The laundromat.

4. (a) Low levels: fighter. High levels: wizard.

   (b) At 5th level, both do the same amount of damage.

   (c) At 8th level.

4.2 Factorial and Exponential Growth

4.2.1 Review of Factorials

Before we compare rates of growth, we need to review a particular notation, the factorial. To denote this, we use an exclamation point, such as

$$3!$$

To compute 3!, we multiply 3 by all positive integers less than 3, so

$$3! = 3 \times 2 \times 1$$

Similarly,

$$5! = 5 \times 4 \times 3 \times 2 \times 1$$

and

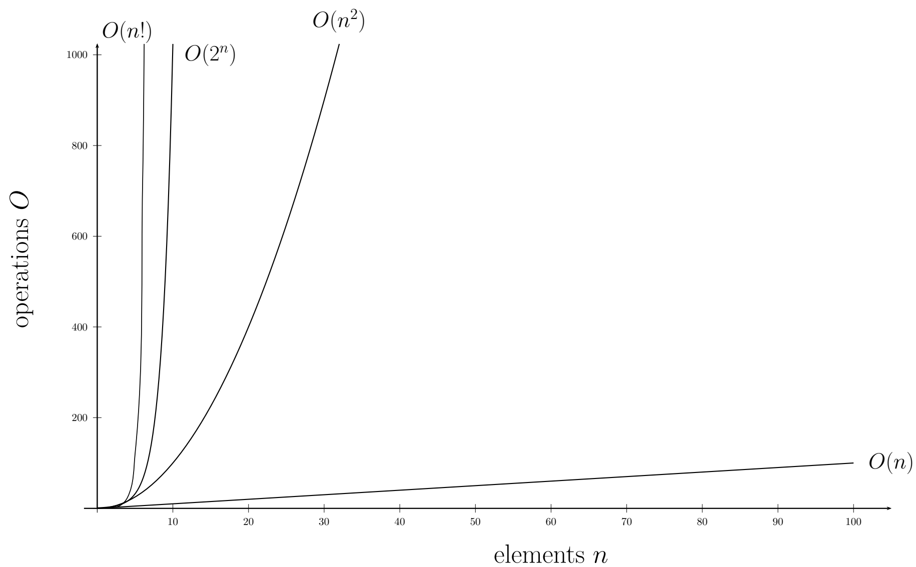
$$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$$

4.2.2 Comparing Rates of Growth

Recall that we are considering how the size of a task n changes the number of steps or operations required to complete the task. To do this, let us examine the following table, which shows how the functions n^2, 2^n, and $n!$ change as n increases.

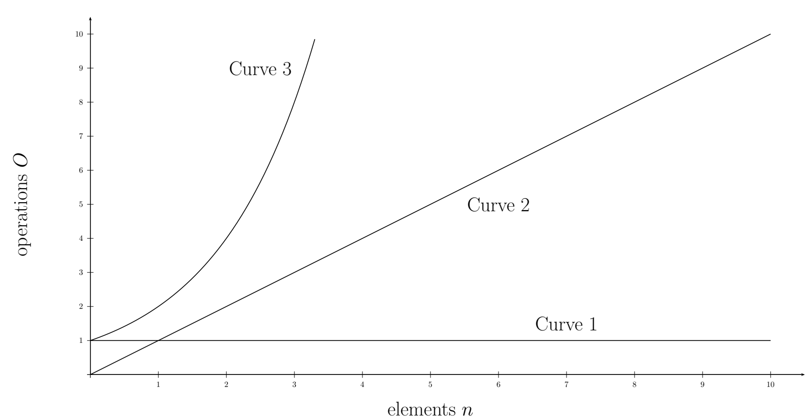
	polynomial	exponential	factorial
n	n^2	2^n	$n!$
1	1	2	1
2	4	4	2
3	9	8	6
4	16	15	24
5	25	32	120
10	100	1024	3628800
100	10000	1.267×10^{30}	9.33×10^{157}

From the table, we can see that although the polynomial n^2 is growing quite fast as n increases, the exponential 2^n is growing faster still. But neither the polynomial or the exponential have the same explosive increase as the factorial function $n!$. We can see these patterns clearly in the following graph.



For this application course, the emphasis is to know the shape of each of the Big O curves, to be able to rank them from slowest growing to fastest, and given multiple curves be able to tell which one is most efficient as n gets large, as in the following example.

> **Example:** Consider the graph below with Curves 1, 2, and 3. Match the Big O notation with its corresponding curve on the graph.



(a) $O(n^2)$ __________

(b) $O(1)$ __________

(c) $O(n)$ __________

Answer:

(a) Curve 3

(b) Curve 1

(c) Curve 2

4.2.3 Why We Omit Coefficients

A good question is why do we omit the coefficients when writing Big O? We had said in the previous section that if a function is equal to a multiple of the amount of data n, then we drop the coefficient and write the Big O notation as $O(n)$. Let's take a look at an extreme example, where our functions are $n!$, $2n!$, and $1000n!$.

	polynomial	exponential	factorial
n	$n!$	2n!	1000n!
1	1	2	1000
2	2	4	2000
3	6	12	6000
4	24	48	24000
5	120	240	120000
10	3628800	7257600	3628800000
100	9.3×10^{157}	1.9×10^{158}	9.3×10^{160}

We can see from the table that the coefficient makes a big difference in the entries when n is 10 or lower. But the moment that we have $n = 100$, then $n!$ itself is so large that the coefficient doesn't really make that big a difference. Do we really care that we have 10^{157} versus 10^{160}? Those numbers are just impossibly large and the difference in coefficients just gets lost in the noise.

4.2.4 Polynomial versus Exponential versus Factorial

Another good question is what happens if we are interested in n^3, not just n^2. How about 3^n or 5^n instead of 2^n? Does this make a different in terms of the growth of that function?

Let's once again look at a table.

	polynomial			exponential			factorial
n	n^2	n^3	n^4	2^n	3^n	4^n	$n!$
1	1	1	1	2	3	4	1
2	4	8	16	4	9	16	2
3	9	27	81	8	27	64	6
4	16	64	256	16	81	256	24
5	25	125	625	32	243	1024	120
10	100	1000	10000	1024	59049	1048576	3628800
100	10^4	10^6	10^8	1.3×10^{30}	5.2×10^{47}	1.6×10^{60}	9.3×10^{157}

Hopefully it's clear from the table that once n gets large (and 100 items is still not a very big task in many circumstances!), that although there are differences in the numbers you get from the different polynomials (10^4, 10^6, 10^8), all of these numbers are orders of magnitude away from the exponential numbers (10^{30}, 10^{47}, 10^{60}). And both of these sets of numbers are dwarfed in comparison with the growth of the factorial (10^{157}).

Basically, if you have any polynomial n^c, where c is a positive constant, it will have a smaller growth pattern than any exponential b^n, where b is a positive constant. For small values of n, that may not yet be the case, but as n gets sufficiently large, the exponential will always get larger than the polynomial eventually.

4.2.5 Big O for Sums of Different Functions

What do you do if you are considering a procedure where the number of steps is $2^n + n^2 + n$. What would Big O be for this sum? Once again, let's look at a table:

n	2^n	n^2	$2^n + n^2 + n$
1	2	1	4
2	4	4	10
3	8	9	20
4	16	16	36
5	32	25	62
10	1024	100	1134
100	1.3×10^{30}	10000	1.3×10^{30}

By the time $n = 100$, which is the term that matters? Clearly $2^n = 1.3 \times 10^{30}$ is so much greater than either $n^2 = 10000$ or $n = 100$ that $2^n + n^2 + n$ is not that much different in value than just 2^n. For large n, those two functions are not identical, true, but their values are so similar that the smaller terms can be safely ignored for most applications.

Our method, then, is to locate in the sum the term that grows the fastest. Then remove any coefficients, and what remains is the order of Big O.

Example: Consider procedures where the number of operations needed for a task of size n is given below. Find Big O for each procedure.

(a) $7n + 3n^2 + 3(2^n)$

(b) $7n + 2^n$

(c) 5

(d) 5!

(e) $35n(n + 1)$

Answer:

(a) Of $O(n)$,$O(n^2)$,and $O(2^n)$, the one with the fastest growth is $O(2^n)$. We drop the leading coefficients.

(b) $O(2^n)$

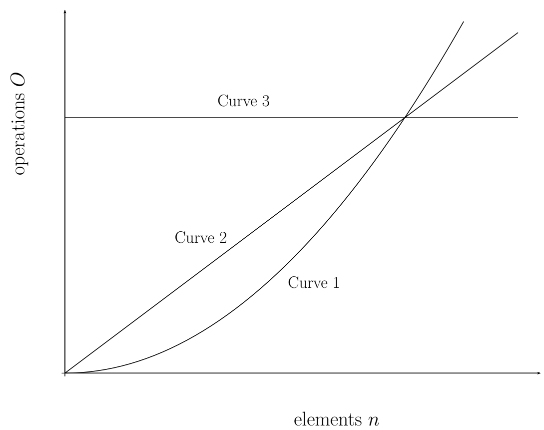
(c) $O(1)$

(d) $O(1)$ - don't be fooled, 5! is just a constant

(e) $35n(n + 1) = 35n^2 + 35n$, so $O(n^2)$

Exercises for Section 4.2

1. Match the Big O notation with its corresponding curve on the graph.

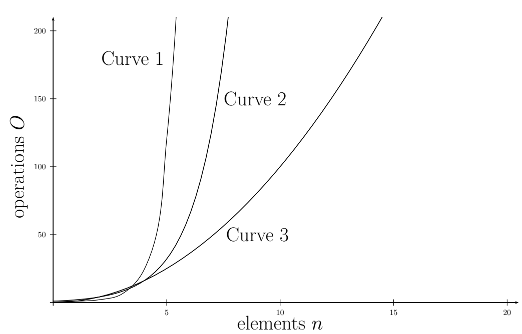


   (a) $O(n^2)$ __________

   (b) $O(1)$ __________

   (c) $O(n)$ __________

2. Match the Big O notation with its corresponding curve on the graph.



   (a) $O(n^2)$ __________

   (b) $O(n!)$ __________

   (c) $O(2^n)$ __________

3. For a task of size n, Program A will always take one million steps to run and Program B will take $5n^2$ steps to run. Indicate whether the

following statements are true or false.

(a) For small n, Program B will run faster than Program A.

(b) For large n, Program B will run faster than Program A.

(c) Program A should always take the same amount of time to run.

(d) If you know the size of the task, you can choose whether Program A or Program B will be more efficient.

(e) If you do not know the size of task, Program B is a good choice because most of the time it will finish faster than Program A.

4. For each of the following procedures, the number of operations needed for a task of size n is given below. Find Big O for each procedure.

   (a) $n^2 + 2n + 3n!$

   (b) $7n + 5$

   (c) 50

   (d) $20n^2 + 40(2^n)$

Answers to Section 4.2 Exercises

1. (a) Curve 1
   (b) Curve 3
   (c) Curve 2
2. (a) Curve 3
   (b) Curve 1
   (c) Curve 2
3. (a) true
   (b) false
   (c) true
   (d) true
   (e) false
4. (a) $O(n!)$
   (b) $O(n)$
   (c) $O(1)$
   (d) $O(2^n)$

4.3 Logarithmic Growth

4.3.1 Growth of $O(\log n)$

Suppose you need to find the position of a particular entry in an ordered list.

Example: Consider the following list of fruit which is in alphabetical order:

apple, banana, grape, orange, peach, pear, plum

Where in the list is the entry **grape**?

Answer:

Method #1: Linear search

Start at the first one on the list and look at each entry in order from left to right until you get to the entry of interest.

This method is $O(n)$. It has the advantage that it is very straightforward to understand, but if you double the size of the list, you will double the number of operations you need to perform in the case that the entry that you want is at the bottom of the list.

Method #2: Binary search

The idea behind this method is that in each step, you cut the length of the list in half when looking for your entry. Let's look at how that works for our list of fruit.

For our first step, the entry halfway through the list is **orange**.

apple, banana, grape, <u>orange</u>, peach, pear, plum

Now, **orange** is not what we want; we want **grape**. And **grape** comes alphabetically before **orange** , so we want the left part of the list, which is

apple, banana, grape

We now repeat the previous steps. The middle entry is **banana**, and **grape** comes after **banana** alphabetically, so we take the list to the right of **banana**, which is

grape

We've now found the entry of interest.

Method #2 may seem much more cumbersome than Method #1, but it has one huge advantage in that it is far more efficient. For example, if your list has one million entries, you need a maximum of 20 searches to find your entry of interest.

This may seem unbelievable, but take a look at the following table for the worst case scenario. After the first step, we have cut the list in half (unless of course the middle entry in the list is the one we want, in which case we are done). We are now down to only 500 000 entries from our starting list of 1 000 000. After the second step, we have only 250 000 entries left. If we keep dividing the list in two relentlessly (and rounding up, because we can't have 0.5 of an entry), in the worst case scenario we take only twenty steps to get to a list that is one item long, which must be the item that we are looking for.

step	size of list	step	size of list
1	500000	11	489
2	250000	12	245
3	125000	13	123
4	62500	14	62
5	31250	15	31
6	15625	16	16
7	7813	17	8
8	3907	18	4
9	1954	19	2
10	977	20	1

Essentially, you are solving

$$2^n = 1000000$$

which requires a new function called a logarithm.

4.3.2 What is a logarithm?

Consider the equation

$$2^3 = 8$$

If we want to write an equivalent statement using logarithms, then we take the base 2 and move it to the other side of the equation and it is then the base of the logarithm:

$$3 = \log_2 8$$

The logarithm asks "what exponent on the base 2 gives the number 8?" and since we know that 2^3 equals 8, the answer is 3.

> **Example:** Evaluate (find the value of) the following expressions containing logarithms.
>
> (a) $\log_2 4$
>
> (b) $\log_2 \frac{1}{2}$
>
> (c) $\log_{10} 1000$
>
> (d) $\log_{10} 1$
>
> (e) $\log_{10} 0$
>
> Answer:
>
> (a) 2 because $2^2 = 4$
>
> (b) -1 because $2^{-1} = \frac{1}{2}$
>
> (c) 3 because $10^3 = 1000$
>
> (d) 0 because $10^0 = 1$
>
> (e) this is undefined because there is no real number x for which $10^x = 0$

In math and most fields in science and technology, if the base is not specified, then it is base 10. So

$$\log 1000 = \log_{10} = 3$$

On a calculator, then the "log" button uses base 10 and the "ln" button (for "natural log) uses base e. The number e is irrational, like π, and is equal to

$$e = 2.718281828459045\ldots$$

Unfortunately, computing does not use base 10 as the default. In most if not all computing languages, the default is base e. In Java, for example, to compute a logarithm using base e, you would use the function `Math.log(value)` to find the natural logarithm (log base e) of a particular value, and `Math.log10(value)` to find the logarithm of that value in base 10.

If you want to calculate the logarithm of any other base, you can use the following relationship.

$$\log_2 x = \frac{\log x}{\log 2}$$

Strangely enough this relationship holds whether you are using a calculator where the default is base 10 or a computing program in which the base is e. (You may have seen this relationship proven if you have taken a pre-calculus course.)

In our initial example, we were trying to solve $2^n = 1000000$. Then

$$n = \log_2 1000000 = \frac{\log 1000000}{\log 2} \approx 19.932$$

Now we can see why our binary search algorithm had a maximum of 20 searches to find any value in a list of one million items, and why it has $O(\log n)$.

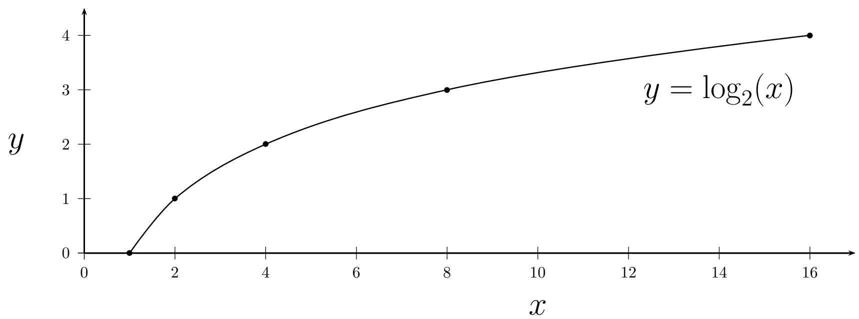
4.3.3 Graph of $y = \log x$)

What does the graph of $O(\log n)$ look like?

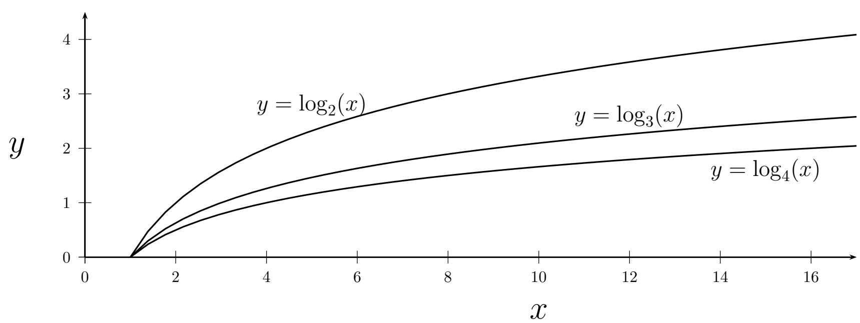
Let's look at the following table.

n	$\log_2 n$
1	0
2	1
4	2
8	3
16	4

If we do a sketch of these values, it looks like this:



What about other bases? What do the logarithmic curves of other bases look like? The following graph shows the curves $y = \log_2(x)$, $y = \log_3(x)$, and $y = \log_4(x)$.

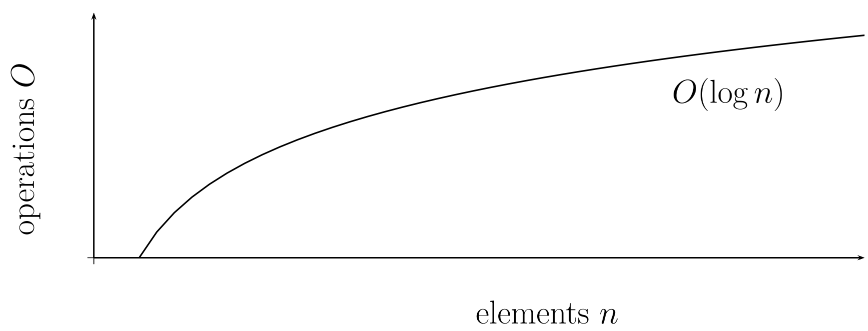


In the above graph you can see that this family of curves all have the same characteristic shape, where the curves are always increasing but the slope gets smaller as the x-value increases. Don't be fooled, though! The curves do not have an upper limit: if you pick any constant value of y, the logarithm curve will always grow above that value of x gets sufficiently large.

In the previous section, we found that the functions $y = 2^x$, $y = 3^x$, and $y = 4^x$ had similar growth patterns, and when studying Big O we group them all together under "exponential growth". Similarly we group all "logarithmic growth" functions together and we don't worry overmuch about which base is being used.

4.3.4 Growth of $O(\log n)$

In summary, the graph of logarithmic Big O is shown below.

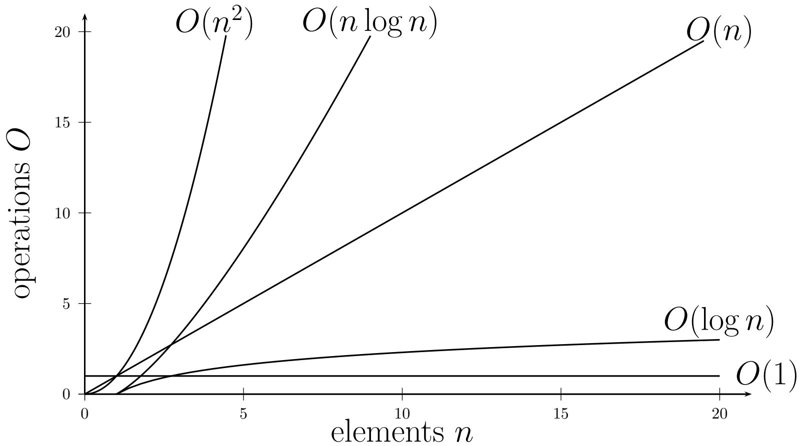


4.3.5 Growth of $O(n \log n)$

There's one other related function that we should also look at, $O(n \log n)$. Sometimes this is called *linearithmic growth.* Let's look at the following table. The values for the logarithms have been rounded to one decimal place.

n	$\log n$	$n \log n$	n^2
1	0	0	1
2	0.3	0.6	4
5	0.7	3.5	25
10	1.0	10.0	100
20	1.3	26.0	400
50	1.7	84.9	2 500
100	2.0	200.0	10 000
200	2.3	460.2	40 000
500	2.7	1349.5	250 000
1000	3.0	3000.0	1 000 000

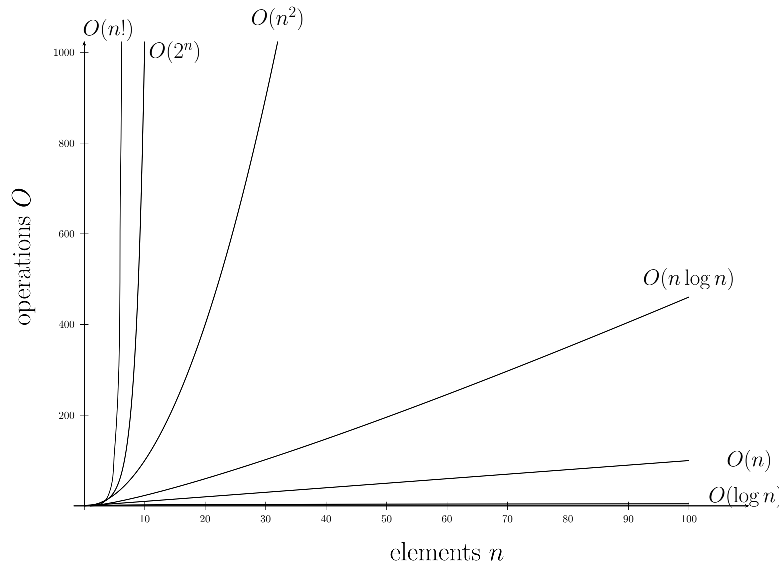
If we do a sketch of these values, it looks like this:



You can see that the graph is quite messy close to the origin, where many of the curves cross each other. However, once n starts getting large, we can see that the $O(n \log n)$ graph lies between the linear $O(n)$ line and the polynomial $O(n^2)$ curve. Although it is hard to see from the scale of this graph, the $O(n \log n)$ curve is not a straight line: it is slightly curved upwards. Having a Big O growth pattern of $O(n \log n)$ is therefore more efficient than polynomial for large n, but not as good as linear growth.

4.3.6 Summary

The full sketch of all Big O rates of growth that we have studied looks like this:



Because of the scale of this graph, you may be able to see the slight curve in $O(n \log n)$, but $O(\log n)$ looks deceptively like a straight line. If you were to change the scale to zoom in on that region of the graph, then $O(\log n)$ would look more like the characteristic curve that we have studied earlier. The scale of this particular graph was chosen to emphasize that the factorial, exponential, and polynomial curves have much larger growth patterns than any of $O(n \log n)$, $O(n)$, and $O(\log n)$. And the most efficient curve of all, of course, is $O(1)$, which we cannot see on this graph at all due to the scale (it would basically lie right on top of the x-axis).

Exercises for Section 4.3

1. Suppose you are trying to find an entry in an ordered list. You try two different methods:

   - Method 1: You start at the beginning of the list and go down until you find the entry you want. This has $O(n)$.

   - Method 2: You go to the halfway point and deterine whether the entry of interest is above or below the that middle entry. Then divide that part of the list in half and check the halfway point. Repeat until you've found the entry of interest. This is called a binary search and has $O(\log n)$.

   Answer the following questions about the above scenario.

   (a) If the list has only 10 items and you are not using a computer for this task, then the most efficient method is probably __________.

   (b) If the list is very long, then the most efficient method is definitely __________.

   (c) For method 1, the best case scenario is that the entry you want is located in the following place:

   top / middle / bottom of the list

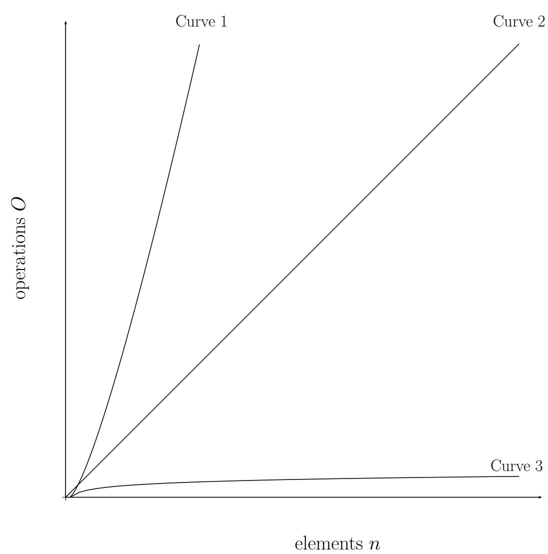
   (d) For method 1, the worst case scenario is that the entry you want is located in the following place:

   top / middle / bottom of the list

   (e) For method 2, the best case scenario is that the entry you want is located in the following place:

   top / middle / bottom of the list

2. Evaluate the following logarithms. Give exact answers.
   (a) $\log_4(16)$
   (b) $\log_{10}(10^6)$
   (c) $\log_{10}(10)$
   (d) $\log_2(256)$
3. Match the Big O notation with its corresponding curve on the graph. Curve 2 is a straight line.



   (a) $O(\log n)$ __________
   (b) $O(n \log n)$ __________
   (c) $O(n)$ __________

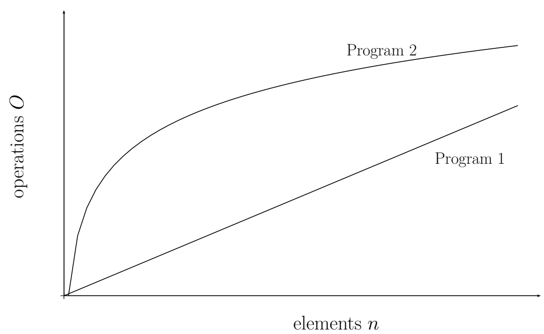
4. Indicate whether the following statements about the $O(\log n)$ curve are true or false.

   (a) If n gets large enough, the curve of $O(\log n)$ will eventually curve downward.

   (b) If n gets large enough, the curve of $O(\log n)$ will reach a certain value and stay there.

   (c) No matter how big n is, the curve of $O(\log n)$ will always increase.

5. This graph shows the number of operations O required to complete a task of size n elements for Programs 1 and 2, where Program 1 is a straight line and Program 2 is a curved line.



   Indicate whether the following statements are true or false.

   (a) Program 1 could be $O(n \log n)$.

   (b) Program 2 could be $O(n \log n)$.

   (c) Program 2 could be $O(\log n)$.

   (d) For large n, Program 1 will finish faster because the line for Program 1 is below the line for Program 2 at the right-hand side of the graph.

6. For each of the following procedures, the number of operations needed for a task of size n is given below. Find Big O for each procedure.

   (a) $n^2 + 2n \log n + 3 \log n$

   (b) $7n + 9n \log n$

   (c) $7 + 2 \log n$

(d) $\log n + 3n$

(e) $n \log n + 3n!$

(f) $(n+1)\log n$

Answers to Section 4.3 Exercises

1. (a) Method 1
   (b) Method 2
   (c) top
   (d) bottom
   (e) middle
2. (a) $\log_4(16) = 2$
   (b) $\log_{10}(10^6) = 6$
   (c) $\log_{10}(10) = 1$
   (d) $\log_2(256) = 8$
3. (a) Curve 3
   (b) Curve 1
   (c) Curve 2
4. (a) false
   (b) false
   (c) true
5. (a) false, $O(n \log n)$ has a slight curve to it and the question said that Program 1 is a straight line
   (b) false
   (c) true
   (d) false
6. (a) n^2
   (b) $n \log n$
   (c) $\log n$
   (d) n
   (e) $n!$
   (f) $n \log n$

Mixed Practice

1. You need to buy a lawnmower, and you have researched the following options. You could get a gas mower for around \$250, an electric mower for \$200, or a robot mower for \$1300.

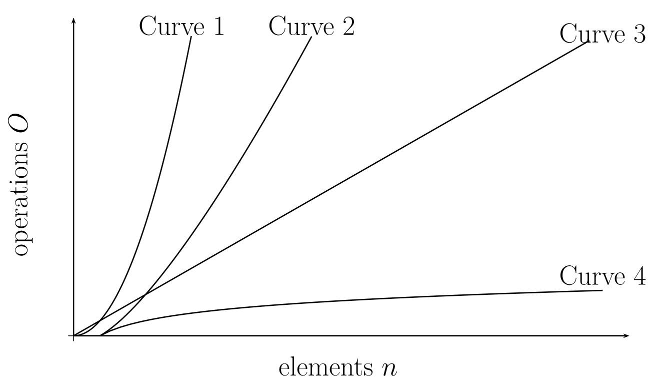
   Indicate whether the following statements are true or false.

   (a) Which mower is the best choice depends on what your priorities are.

   (b) The larger the lawn is, the longer it will take you to mow if you choose either the gas or electric mower options.

   (c) If your lawn is twice as big as your neighbour's, it will take you twice as long as your nieghbour to mow it with either the gas or electric mowers (provided of course that your neighbour is using the same type of mower), so the time to mow would probably be $O(n)$.

2. Match the Big O notation with its corresponding curve on the graph. Please note that curve 3 is a straight line.



   (a) $O(n \log n)$

   (b) $O(\log n)$

   (c) $O(n^2)$

   (d) $O(n)$

3. Evaluate the following logarithms. Give exact answers.

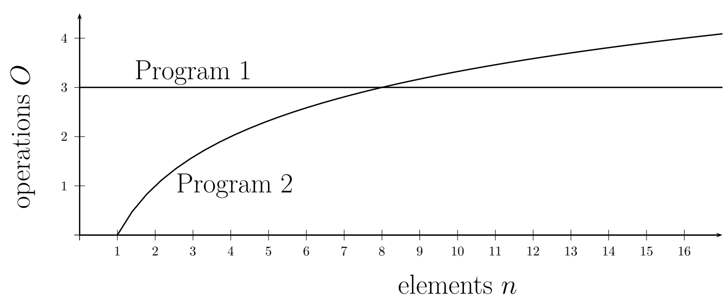
   (a) $\log_3(81)$

(b) $\log_{10}(0.01)$

(c) $\log_2(1)$

(d) $\log_4(4)$

4. The following graph shows the number of operations O required to complete a task of size n for Programs 1 and 2. The number of operations required for Program 1 is a constant, so Program 1 is a horizontal straight line.



Indicate whether the following statements are true or false.

(a) It's possible that for a certain value of n, the two programs are equally efficient.

(b) Program 2 is a better choice than Program 1 for some circumstances.

(c) If Program 2 is $O(\log n)$, then for large values of n it could curve downwards and become more efficient than Program 1.

5. If you look up algorithms on how to search a list, you will find that in terms of operations, a linear search has $O(n)$ while a binary search has $O(\log n)$.

Based only on this information, which method is more efficient for large values of n? Indicate the correct choice.

(a) linear seach

(b) binary search

(c) they both have the same efficiency

Why?

(a) Because n grows faster than $\log n$ and bigger is better.

(b) There is not enough information to decide.

(c) Because $\log n$ grows slower than n and fewer operations means that the program will run faster.

(d) Because n and $\log n$ grow at the same rate.

6. For each of the following procedures, the number of operations needed for a task of size n is given below. Find Big O for each procedure.

   (a) $2^n + 5n!$

   (b) $\log n + n$

   (c) $3 + 2 + 1!$

   (d) $n(n + \log n + 1)$

   (e) $n \log n + 2n$

Answers

1. (a) true

   (b) true

   (c) true

2. (a) Curve 2

   (b) Curve 4

   (c) Curve 1

   (d) Curve 3

3. (a) $\log_3(81) = 4$

   (b) $\log_{10}(0.01) = -2$

   (c) $\log_2(1) = 0$

   (d) $\log_4(4) = 1$

4. (a) true

   (b) true

   (c) false

5. binary search, because $\log n$ grows slower than n and fewer operations means that the program will run faster.

6. (a) $O(n!)$

   (b) $O(n)$

   (c) $O(1)$

   (d) $O(n^2)$

   (e) $O(n \log n)$

List of Symbols

p, q, r	logical propositions
$\sim p$	negation of p
$\wedge$	and
$\vee$	inclusive or
$\oplus$	exclusive or
$\Leftrightarrow$	logically equivalent
$\rightarrow$	conditional
$\leftrightarrow$	biconditional
$\overline{A}$	negation of A (Boolean)
$\cdot$	and (Boolean)
$+$	inclusive or (Boolean)
a_n	n^{th} term of a sequence

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