8.6 Double and Half Angle Formulas

Double Angle Formulas

$$\sin 2\theta = 2\sin\theta\cos\theta$$

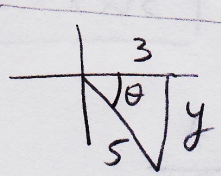
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\cos 2\theta = 1 - 2\sin^2\theta$$

Ex: Find $\tan 2\theta$ if $\cos\theta = \frac{3}{5}$ and $\sin\theta < 0$

Find $\tan\theta$:

S	A
T	C



$$\cos\theta = \frac{3}{5} = \frac{x}{r}$$

$$y^2 + 3^2 = 5^2$$
$$y^2 = 16$$
$$y = \pm 4$$
$$y = -4$$

$$\tan\theta = \frac{y}{x} = \frac{-4}{3}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{\frac{-8}{3}}{\left(1-\frac{16}{9}\right)} \cdot \frac{9}{9} = \frac{-24}{9-16} = \frac{-24}{-7} = \frac{24}{7}$$

Ex: Show that $\csc 2\theta = \frac{1}{2}\sec\theta\csc\theta$

$$\csc 2\theta = \frac{1}{\sin 2\theta}$$
$$= \frac{1}{2\sin\theta\cos\theta}$$
$$= \frac{\csc\theta\sec\theta}{2}$$
$$= \frac{1}{2}\sec\theta\csc\theta \checkmark$$

Ex: Solve $\sin\theta\cos\theta = \frac{-1}{4}$, $0 \le \theta < \pi$

$$2\sin\theta\cos\theta = \frac{-1}{2}$$
$$\sin 2\theta = \frac{-1}{2}$$

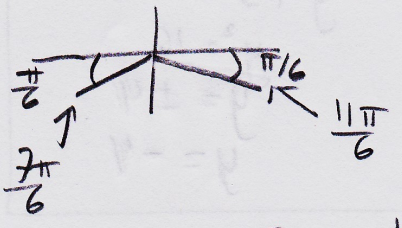
Let $x = 2\theta$ $\quad \boxed{\sin x = \frac{-1}{2}}$

$0 \le 2\theta < 2\pi$

$\boxed{0 \le x < 2\pi}$

S	A
T	C

$\sin x = \frac{-1}{2} \Rightarrow$ reference angle $x = \frac{\pi}{6}$



$$x = \frac{7\pi}{6}, \frac{11\pi}{6} = 2\theta$$

$$\theta = \frac{7\pi}{12}, \frac{11\pi}{12}$$

Ex: Find a formula for $\sin 3\theta$ in terms of $\sin\theta$ and $\cos\theta$.

$$\begin{aligned}\sin 3\theta &= \sin(2\theta+\theta)\\ &= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta\\ &= 2\sin\theta\cos\theta\cos\theta + (\cos^2\theta - \sin^2\theta)\sin\theta\\ &= 2\sin\theta\cos^2\theta + \cos^2\theta\sin\theta - \sin^3\theta\\ &= 3\sin\theta\cos^2\theta - \sin^3\theta\end{aligned}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

Ex: Write $\cos^4\theta$ in terms of $\cos 2\theta$ and $\cos 4\theta$

$$\begin{aligned}\cos^4\theta &= (\cos^2\theta)^2\\ &= \left(\frac{1+\cos 2\theta}{2}\right)^2\\ &= \frac{1+2\cos 2\theta+\cos^2 2\theta}{4}\\ &= \frac{1}{4} + \frac{\cos 2\theta}{2} + \frac{\cos^2 2\theta}{4}\end{aligned}$$

$$= \frac{1}{4} + \frac{\cos 2\theta}{2} + \frac{1}{4}\left(\frac{1+\cos 4\theta}{2}\right)$$

$$= \frac{1}{4} + \frac{\cos 2\theta}{2} + \frac{1}{8} + \frac{\cos 4\theta}{8}$$

$$= \frac{3}{8} + \frac{\cos 2\theta}{2} + \frac{\cos 4\theta}{8}$$

Half Angle Formulas

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}$$

$$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$$

Ex: Find the exact value of:

a) $\sin\frac{\pi}{12}$

S | (A)
T | C

$\sin\frac{\pi}{12} > 0$

$$\sin\frac{\pi}{12} = +\sqrt{\frac{1-\cos\frac{\pi}{6}}{2}}$$

$$= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} \quad \times \frac{2}{2} \text{ under root}$$

$= \sqrt{\frac{2-\sqrt{3}}{4}}$ or $\frac{\sqrt{2-\sqrt{3}}}{2}$

b) $\sin\left(\frac{-\pi}{12}\right)$

$= -\sin\frac{\pi}{12}$

$= -\frac{\sqrt{2-\sqrt{3}}}{2}$

Ex: Find $\cos\frac{\alpha}{2}$ if $\cos\alpha = -\frac{4}{5}$, $\pi < \alpha < \frac{3\pi}{2}$

$\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}$

(S)	A
T	C

$\cos\frac{\alpha}{2} < 0$

$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}}$

$= -\sqrt{\frac{1-\frac{4}{5}}{2}}$ $\times \frac{5}{5}$ under root

$= -\sqrt{\frac{5-4}{10}}$

$= -\sqrt{\frac{1}{10}}$

$= -\frac{1}{\sqrt{10}}$

$= -\frac{\sqrt{10}}{10}$