8.5 Sum and Difference Formulas

$$\boxed{\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \end{aligned}}$$

Ex: Find exact values

a) $\cos 15°$

$\boxed{15° = 45° - 30°}$

$$\begin{aligned} &= \cos(45° - 30°) \\ &= \cos 45° \cos 30° + \sin 45° \sin 30° \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\ &= \frac{\sqrt{6} + \sqrt{2}}{4} \end{aligned}$$

b) $\sin 75° \cos 15° - \cos 75° \sin 15°$

$$\begin{aligned} &= \sin(75° - 15°) \\ &= \sin 60° \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

$$\boxed{\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\\ \tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\end{aligned}}$$

Ex: Exact value of $\tan\frac{5\pi}{12}$?

$$\boxed{\begin{aligned}\frac{5\pi}{12} &= \frac{2\pi}{12}+\frac{3\pi}{12}\\ &= \frac{\pi}{6}+\frac{\pi}{4}\end{aligned}}$$

$$\boxed{\tan\frac{\pi}{6}=\frac{\sqrt{3}}{3} \quad \tan\frac{\pi}{4}=1}$$

$$\begin{aligned}\tan\frac{5\pi}{12} &= \tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)\\ &= \frac{\tan\frac{\pi}{6}+\tan\frac{\pi}{4}}{1-\tan\frac{\pi}{6}\tan\frac{\pi}{4}}\\ &= \frac{\frac{\sqrt{3}}{3}+1}{1-\frac{\sqrt{3}}{3}\cdot 1}\cdot\frac{3}{3}\\ &= \frac{\sqrt{3}+3}{3-\sqrt{3}}\end{aligned}$$

Ex: Show that $\tan\left(\theta+\frac{\pi}{2}\right)=-\cot\theta$

Caution: $\tan\frac{\pi}{2}$ is undefined
Can't use $\tan(\alpha+\beta)$ formula

$$\tan\left(\theta + \frac{\pi}{2}\right) = \frac{\sin\left(\theta + \frac{\pi}{2}\right)}{\cos\left(\theta + \frac{\pi}{2}\right)}$$

$$= \frac{\sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2}}{\cos\theta\cos\frac{\pi}{2} - \sin\theta\sin\frac{\pi}{2}}$$

$$\boxed{\cos\frac{\pi}{2} = 0 \quad \sin\frac{\pi}{2} = 1}$$

$$= \frac{\cos\theta}{-\sin\theta}$$

$$= -\cot\theta$$

Ex: Show $\dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \dfrac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta}$

$$\frac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta} = \frac{\left(\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}\right)}{\left(\frac{\sin\alpha}{\cos\alpha} - \frac{\sin\beta}{\cos\beta}\right)} \cdot \frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}$$

$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\cos\beta - \cos\alpha\sin\beta}$$

$$= \frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$$

<u>Ex</u>: Let $0 < \alpha, \beta < \frac{\pi}{2}$

Given $\cos\alpha = \frac{\sqrt{3}}{2}$ and $\sin\beta = \frac{1}{4}$,

find $\sin(\alpha+\beta)$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$\boxed{\cos\alpha = \frac{\sqrt{3}}{2}}$$

$$\sin^2\alpha + \cos^2\alpha = 1$$
$$\sin^2\alpha + \frac{3}{4} = 1$$
$$\sin^2\alpha = \frac{1}{4}$$
$$\sin\alpha = \pm\frac{1}{2}$$

α in QI $\rightarrow$ $\boxed{\sin\alpha = \frac{1}{2}}$

Similarly: $\boxed{\sin\beta = \frac{1}{4}}$

implies $\boxed{\cos\beta = \frac{\sqrt{15}}{4}}$

$$\sin(\alpha+\beta) = \frac{1}{2}\frac{\sqrt{15}}{4} + \frac{\sqrt{3}}{2}\cdot\frac{1}{4}$$
$$= \frac{\sqrt{15}+\sqrt{3}}{8}$$

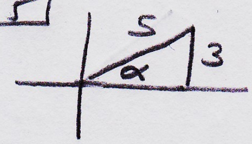
Ex: Find $\sin\left(\sin^{-1}\frac{3}{5} - \cos^{-1}\left(\frac{-4}{5}\right)\right)$

Let $\alpha = \sin^{-1}\frac{3}{5}$ and $\beta = \cos^{-1}\left(-\frac{4}{5}\right)$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\alpha = \sin^{-1}\frac{3}{5}$

$\boxed{\sin\alpha = \frac{3}{5}}$



$x^2 + 3^2 = 5^2$

$x^2 = 16$

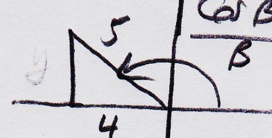
$x = \pm 4$

$x = 4$

$\boxed{\cos\alpha = \frac{x}{r} = \frac{4}{5}}$

$\beta = \cos^{-1}\left(\frac{-4}{5}\right)$

$\boxed{\cos\beta = \frac{-4}{5}}$



$(x = -4)$

$4^2 + y^2 = 5^2$

$y^2 = 9$

$y = \pm 3$

$y = 3$

$\boxed{\sin\beta = \frac{y}{r} = \frac{3}{5}}$

$\sin(\alpha-\beta) = \frac{3}{5}\left(\frac{-4}{5}\right) - \frac{4}{5}\cdot\frac{3}{5}$

$= \frac{-24}{25}$