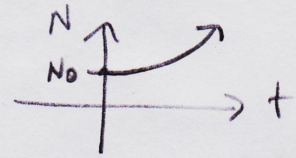
6.8 Exponential Growth and Decay; Newton's Law

Exponential Growth

$$N = N_0 e^{kt}$$

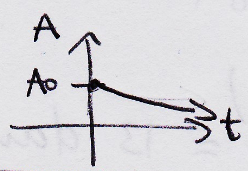
N: amount at time t
N_0: initial amount
k: growth rate $(k > 0)$



Exponential Decay

$$A = A_0 e^{kt}$$

A: amount remaining at time t
A_0: initial amount
k: decay rate $(k < 0)$



Ex: Exponential Growth

Bacteria growth is given by $N = 6.0 e^{0.038t}$
where N: grams and t: days
Find:

a) initial mass

Sub $t=0$: $\quad N = 6.0e^{0} = 6.0\text{ g}$

b) growth rate

$k = 0.038$ or 3.8% per day

(indicates how fast exponent changes)

c) mass after 10 days

$t=10$: $\quad N = 6.0e^{0.38}$

$\approx 8.8\text{ g}$

d) time when mass is 10 g

$N=10$: $\quad 10 = 6.0e^{0.038t}$

$$\frac{10}{6} = e^{0.038t}$$

$$\ln\left(\frac{10}{6}\right) = \ln e^{0.038t}$$

$$\ln\left(\frac{10}{6}\right) = 0.038t$$

$$t = \frac{\ln\left(\frac{10}{6}\right)}{0.038} \approx 13\text{ days}$$

Half-life: Time for 50% of a substance to decay

Ex: Radioactive Decay

Half-life of Carbon-14 is 5600 years.
How long for 70% of a sample to decay?

$$\boxed{A = A_0 e^{kt}} \qquad t = \text{years}$$

Sub $A = 0.5A_0$ (half of initial amount) ;
$t = 5600$

$$0.5A_0 = A_0 e^{k(5600)}$$
$$0.5 = e^{5600k}$$
$$\ln 0.5 = 5600k$$
$$k = \frac{\ln 0.5}{5600}$$

$$\boxed{A = A_0 e^{\frac{\ln 0.5}{5600} t}}$$

70% decayed = 30% remaining

Sub $A = 0.3A_0$:
$$0.3A_0 = A_0 e^{\frac{\ln 0.5}{5600} t}$$
$$0.3 = e^{\frac{\ln 0.5}{5600} t}$$
$$\ln 0.3 = \frac{\ln 0.5}{5600} t$$

$$t = \frac{\ln 0.3}{\ln 0.5} \times 5600 \approx 9727 \text{ years}$$

Newton's Law of Cooling

$$u = T + (u_0 - T)e^{kt} \qquad (k<0)$$

u: object's temperature at time t

u_0: object's initial temp.

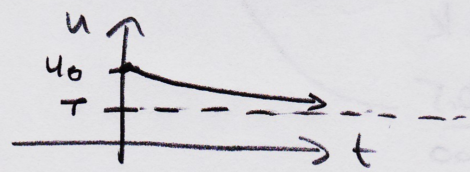
T: constant environmental temp.

"A cooling object's temperature decays exponentially"

Cooling object $\Rightarrow k<0$

As $t \to \infty$ $\quad e^{kt} \to 0$

As $t \to \infty$ $\quad u = T + (u_0 - T)e^{kt} \to T$



Ex: 100°C water cools to 75°C in 5 minutes. Time to reach 50°C? Room temp is 20°C.

$$\boxed{u = T + (u_0 - T)e^{kt}}$$

temp: °C
time: minutes

$u_0 = 100°C$
$T = 20°C$:

$$\boxed{u = 20 + 80e^{kt}}$$

Sub $u = 75°C$, $t = 5$: $75 = 20 + 80e^{k(5)}$

$$55 = 80e^{5k}$$

$$\frac{55}{80} = e^{5k}$$

$$\ln\left(\frac{55}{80}\right) = 5k$$

$$k = \frac{1}{5}\ln\left(\frac{55}{80}\right)$$

$$\boxed{u = 20 + 80e^{\frac{1}{5}\ln\left(\frac{55}{80}\right)t}}$$

$u = 50°C$:

$$50 = 20 + 80e^{kt}$$

$$30 = 80e^{kt}$$

$$\frac{3}{8} = e^{kt}$$

$$\ln\left(\frac{3}{8}\right) = kt$$

$$t = \frac{1}{k}\ln\left(\frac{3}{8}\right)$$

$$= \frac{5}{\ln\left(\frac{55}{80}\right)}\ln\left(\frac{3}{8}\right)$$

≈ 13 minutes