6.7 Financial Models

Compound interest: Interest is regularly added to the balance. Balance and interest then earn interest.

Ex: \$10,000 earns 3% annual interest, compounded quarterly, for 20 years. Value after 20 years?

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

A = future value (\$)
P = present value or principal (\$)
r = annual interest rate
n = # compounding periods per year
t = time (years)

$P = 10{,}000 \quad r = 0.03 \quad n = 4 \quad t = 20$

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$
$$= 10{,}000\left(1 + \frac{0.03}{4}\right)^{80}$$
$$\approx \$18{,}180.44$$

Ex: How much is interest?

$A - P = \$8{,}180.44$

n = # compounding periods per year

As $n \to \infty$, $\left(1+\frac{r}{n}\right)^{nt} \to e^{rt}$

$$\boxed{\text{"Continuous Compounding"} \quad A = Pe^{rt}}$$

__Ex__: \$10,000 earns 3% annual interest, compounded continuously, for 20 years. Value after 20 years?

$P = 10{,}000 \quad r = 0.03 \quad t = 20$

$A = Pe^{rt}$

$= 10{,}000\, e^{0.6}$

$\approx \$18{,}221.19$

Effective rate r_e: interest earned on \$1 after 1 year (stated as a %)

$$\boxed{\begin{array}{ll} r_e = \left(1+\frac{r}{n}\right)^{n} - 1 & \text{Compound interest} \\ r_e = e^{r} - 1 & \text{Continuous Compounding} \end{array}}$$

Why? $r_e = A - P$ when $P = 1$ and $t = 1$

Ex: Find effective rate:

a) 3% compounded monthly

$r = 0.03 \quad n = 12$

$$r_e = \left(1 + \frac{r}{n}\right)^n - 1$$

$$= \left(1 + \frac{0.03}{12}\right)^{12} - 1$$

$$\approx 0.0304$$

$$\boxed{r_e \approx 3.04\%}$$

b) 4% compounded continuously

$$r_e = e^r - 1 \qquad r = 0.04$$

$$= e^{0.04} - 1$$

$$\approx 0.0408$$

$$\boxed{r_e \approx 4.08\%}$$

Ex: Investment earns 4%, compounded monthly. What amount should be deposited today to have \$100,000 in 15 years?

$r = 0.04 \quad n = 12 \qquad A = 100{,}000 \quad t = 15$

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$100{,}000 = P\left(1 + \frac{0.04}{12}\right)^{180}$$

$$P = \frac{100000}{\left(1 + \frac{0.04}{12}\right)^{180}} \approx \$54{,}935.95$$

Ex: Interest is compounded annually. Find the rate that doubles the investment in 8 years.

$n=1 \quad A=2P \quad t=8$

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$2P = P(1+r)^8$$

$$2 = (1+r)^8$$

$$2^{1/8} = 1+r$$

$$r = 2^{1/8}-1$$

$$\approx 0.0905$$

$$\boxed{r \approx 9.05\%}$$

Ex: Investment earns 3%, compounded monthly. How long to double the investment?

$r = 0.03 \quad n=12 \quad A=2P$

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$2P = P\left(1+\frac{0.03}{12}\right)^{12t}$$

$$2 = \left(1+\frac{0.03}{12}\right)^{12t}$$

$$\ln 2 = \ln\left(1+\frac{0.03}{12}\right)^{12t}$$

$$\ln 2 = 12t \ln\left(1 + \frac{0.03}{12}\right)$$

$$t = \frac{\ln 2}{12 \ln\left(1 + \frac{0.03}{12}\right)}$$

$$\approx 23.1 \text{ years}$$