5.5 Real Zeros of a Polynomial

$$\frac{\text{Polynomial}}{\text{Divisor}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}$$

$$\boxed{\text{Polynomial} = \text{Quotient} \cdot \text{Divisor} + \text{Remainder}}$$

Ex: Divide $f(x) = x^3 + 7x^2 - 36$ by $x-5$ and write as $P = Q \cdot D + R$

$$\begin{array}{r l}
 & x^2 + 12x + 60 \quad \longleftarrow \text{quotient} \\
x-5 \,\big)\!\!\! & x^3 + 7x^2 + 0x - 36 \\
- & (x^3 - 5x^2) \\
 & 12x^2 + 0x - 36 \\
- & (12x^2 - 60x) \\
 & 60x - 36 \\
- & (60x - 300) \\
 & 264 \quad \leftarrow \text{remainder}
\end{array}$$

$$\boxed{x^3 + 7x^2 - 36 = (x^2 + 12x + 60)(x-5) + 264}$$

Notice $f(5) = 5^3 + 7(5)^2 - 36$
$= 264$

Remainder Theorem

Let f be a polynomial function. If $f(x)$ is divided by $x-c$ the remainder is $f(c)$.

$x-c$ is a factor of $f(x)$ exactly when $\underbrace{\text{remainder} = 0}_{f(c)=0}$

Factor Theorem

Let f be a polynomial function.
$x-c$ is a factor of $f(x)$ if and only if $f(c)=0$

Means:
If $x-c$ is a factor of $f(x)$ then $f(c)=0$
<u>and</u>
If $f(c)=0$ then $x-c$ is a factor of $f(x)$

<u>Ex</u>: a) Remainder when $f(x) = x^4+17x^3+380$ is divided by $x+3$?

$f(-3) = (-3)^4 + 17(-3)^3 + 380$
$= 2$

b) Is $x+3$ a factor of $f(x)$?

No because remainder $\neq 0$

<u>Zero</u> of a function f: number c that makes $f(c)=0$

<u>Rational Zeros Theorem</u>
Given a polynomial with integer coefficients, each rational zero is of the form
$\dfrac{p}{q}$ p ← divides constant, q ← divides leading coefficient

<u>Ex</u>: List the possible rational zeros of

$$f(x) = 2x^3 - 11x - 7$$

Constant $= -7$

p divides -7

$p: \pm 1, \pm 7$

Leading coefficient $= 2$

q divides 2

$q: \pm 1, \pm 2$

All possibilities for $\frac{p}{q}$:

$$\frac{\pm 1}{\pm 1}, \quad \frac{\pm 7}{\pm 1}, \quad \frac{\pm 1}{\pm 2}, \quad \frac{\pm 7}{\pm 2}$$

$$\boxed{\pm 1, \pm 7, \pm \frac{1}{2}, \pm \frac{7}{2}}$$

Ex: Solve $f(x)=0$ for $f(x)=x^3+5x^2+7x+3$
(find all real zeros)

1) Use Rational Zeros Theorem to find one zero:

$p: \pm 1, \pm 3$

$q: \pm 1$

$\frac{p}{q}: \pm 1, \pm 3$

Test if $f(\#)=0$

$f(1) \neq 0$

$f(-1) = 0$ ✓

-1 is a zero of $f(x)$

2) Use Factor Theorem and divide:

-1 is a zero of $f(x) \Rightarrow x+1$ is a factor

$$f(x) = (x+1)\,g(x)$$

$$\begin{array}{r|l} & x^2+4x+3 \quad \leftarrow g(x) \\ \hline x+1 & x^3+5x^2+7x+3 \\ & -(x^3+x^2) \\ \hline & 4x^2+7x+3 \\ & -(4x^2+4x) \\ \hline & 3x+3 \\ & -(3x+3) \\ \hline & 0 \end{array}$$

$$f(x) = (x+1)(x^2+4x+3)$$

Factor fully $f(x) = (x+1)(x+1)(x+3)$

$= (x+1)^2(x+3)$

Real zeros : $x = -1, -3$

Ex: Factor $f(x) = 2x^4 - x^3 - 11x^2 + 5x + 5$ fully

1) Find one zero using Rational Zeros Theorem :

$p: \pm 1, \pm 5$

$q: \pm 1, \pm 2$

$\frac{p}{q}: \pm 1, \pm 5, \pm \frac{1}{2}, \pm \frac{5}{2}$

Test if $f(\#) = 0$

$f(1) = 0$ ✓

1 is a zero of $f(x)$

2) Use Factor Theorem and divide :

1 is a zero of $f(x) \Rightarrow x-1$ is a factor

$f(x) = (x-1)\,g(x)$

$$\begin{array}{r l} & 2x^3 + x^2 - 10x - 5 \quad \Leftarrow g(x) \\ x-1 & \overline{)\,2x^4 - x^3 - 11x^2 + 5x + 5} \end{array}$$

$f(x) = (x-1)(2x^3 + x^2 - 10x - 5)$

Repeat Step 1 and 2 on new quotient

3) Find a zero of $g(x) = 2x^3 + x^2 - 10x - 5$:

$p: \pm 1, \pm 5$

$q: \pm 1, \pm 2$

$\frac{p}{q}: \pm 1, \pm 5, \pm \frac{1}{2}, \pm \frac{5}{2}$

Test if $g(\#) = 0$

$g(1) \neq 0$

$g(-1) \neq 0$

$\vdots$

$g(\frac{-1}{2}) = 0$

$-\frac{1}{2}$ is a zero of $g(x)$

4) Use Factor Theorem and divide :

$x + \frac{1}{2}$ is a factor of $g(x)$

$g(x) = (x + \frac{1}{2})\, h(x)$

$$\begin{array}{r|l} & 2x^2 - 10 \quad \leftarrow h(x) \\ x + \frac{1}{2} & 2x^3 + x^2 - 10x - 5 \end{array}$$

$$\boxed{\begin{aligned} g(x) &= (x + \tfrac{1}{2})(2x^2 - 10) \\ &= 2(x + \tfrac{1}{2})(x^2 - 5) \end{aligned}}$$

Recall $f(x) = (x-1)\, g(x)$

$= 2(x-1)(x + \frac{1}{2})(x^2 - 5)$

Every polynomial with real coefficients can be factored into linear and/or irreducible quadratic factors

Ex: Find all real zeros of $f(x)$

a) $f(x) = (x-7)(x^2-5)$

linear

irreducible (can't be factored using rational #)

$(x-7)(x^2-5)=0$

$x-7=0$
$x=7$

$x^2-5=0$
$x=\pm\sqrt{5}$

$\boxed{x = 7, \pm\sqrt{5}}$

b) $f(x) = (x+4)(x^2+1)$

$(x+4)(x^2+1)=0$

$x+4=0$
$x=-4$

$x^2+1=0$
no real solution

$\boxed{x=-4}$

c) $f(x) = (x-6)(x+2)(x-2)$

$\boxed{x = 6, \pm 2}$