13.3 Geometric Sequences

A sequence is <u>geometric</u> if $\frac{a_n}{a_{n-1}}$ is constant.

$r = \frac{a_n}{a_{n-1}}$ "the common ratio"

<u>Ex</u>: $\underbrace{3, 6}_{\times 2}, \underbrace{12}_{\times 2}, \underbrace{24}_{\times 2}, \ldots$ is geometric

$r = 2$ details: $r = \frac{a_n}{a_{n-1}}$

$= \frac{6}{3}$

$= 2$

<u>Ex</u>: Is the sequence geometric? If so, find a_1 and r

a) $\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

not geometric

b) $3, \frac{3}{4}, \frac{3}{16}, \ldots$

geometric $a_1 = 3$ $r = \frac{1}{4}$ details: $r = \frac{a_n}{a_{n-1}}$

$= \frac{\left(\frac{3}{4}\right)}{3}$

$= \frac{1}{4}$

$$\boxed{\begin{array}{l} n^{th} \text{ term of a geometric sequence} \\ a_n = a_1 r^{n-1} \end{array}}$$

$$\begin{array}{ccccc} a_1, & a_2, & a_3, & \dots, & a_n \\ & \uparrow & \uparrow & & \uparrow \\ & a_1 r & a_1 r^2 & & a_1 r^{n-1} \end{array}$$

<u>Ex</u>: Find the n^{th} term of $8, \frac{64}{9}, \frac{512}{81}, \dots$

geometric $a_1 = 8$ $r = \frac{8}{9}$

details: $r = \frac{a_n}{a_{n-1}} = \frac{\left(\frac{64}{9}\right)}{8} = \frac{8}{9}$

$$\boxed{\begin{aligned} a_n &= a_1 r^{n-1} \\ &= 8\left(\frac{8}{9}\right)^{n-1} \end{aligned}}$$

<u>Ex</u>: Find a_n for the geometric sequence with $a_2 = 6$ and $a_5 = 48$

$a_2 = 6$ $\quad\quad$ $a_5 = 48$

$\boxed{a_1 r = 6}$ ① $\quad\quad$ $\boxed{a_1 r^4 = 48}$ ②

② ÷ ①: $\frac{a_1 r^4}{a_1 r} = \frac{48}{6}$

$r^3 = 8$

$r = 2$

→ either equation ① $a_1(2) = 6$

$a_1 = 3$

$$a_n = a_1 r^{n-1} = 3(2^{n-1})$$

Sum of the first n terms of a geometric sequence

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

Ex: Find $3 + \frac{3}{2} + \frac{3}{4} + \dots + \frac{3}{512}$

geometric $a_1 = 3$ $r = \frac{1}{2}$ $n = ?$

$$a_n = \frac{3}{512}$$

$$a_1 r^{n-1} = \frac{3}{512}$$

$$3\left(\frac{1}{2}\right)^{n-1} = \frac{3}{512}$$

$$\left(\frac{1}{2}\right)^{n-1} = \frac{1}{512}$$

$$2^{n-1} = 512$$

$$\ln 2^{n-1} = \ln 512$$

$$(n-1)\ln 2 = \ln 512$$

$$n-1 = \frac{\ln 512}{\ln 2} = 9$$

$$n = 10$$

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_{10} = \frac{3\left(1-\left(\frac{1}{2}\right)^{10}\right)}{1-\frac{1}{2}}$$

$$= \frac{3\left(1-\frac{1}{1024}\right)}{\left(\frac{1}{2}\right)} \times \frac{1024}{1024}$$

$$= \frac{3\,(1024-1)}{512}$$

$$= \frac{3069}{512}$$

$1 + 0.1 + 0.01 + 0.001 + \ldots = 1.\overline{1}$

If $-1 < r < 1$, terms get closer and closer to 0
and $S_\infty = a_1 + a_2 + a_3 + \ldots$ is a real #

> If $-1 < r < 1$, $\quad S_\infty = \dfrac{a_1}{1-r}$
>
> If $r \geq 1$ or $r \leq -1$, S_∞ does not exist

Ex: Find $3 + \frac{3}{2} + \frac{3}{4} + \ldots$

geometric $\quad a_1 = 3 \quad r = \frac{1}{2} \quad n = \infty$

$$S_\infty = \frac{a_1}{1-r} = \frac{3}{(1-\frac{1}{2})} = \frac{3}{(\frac{1}{2})} = 6$$

> Note: $S_3 = 3 + \frac{3}{2} + \frac{3}{4} = 5.25$
>
> $S_{10} = \dfrac{3069}{512} \simeq 5.99$ (above)
>
> $S_\infty = 6$ seems reasonable

Ex: Write $0.\overline{8}$ as a fraction

$$0.\overline{8} = 0.888888\ldots$$
$$= 0.8 + 0.08 + 0.008 + 0.0008 + \ldots$$

geometric $a_1 = 0.8$ $r = 0.1$

$$S_\infty = \frac{a_1}{1-r} = \frac{0.8}{1-0.1} = \frac{0.8}{0.9} = \frac{8}{9}$$

$$\boxed{0.\overline{8} = \frac{8}{9}}$$

Recursive definition of a geometric sequence:

$a_n = (a_{n-1})r$, indicate a_1
$n \geq 2$

Ex: $7, \frac{7}{5}, \frac{7}{25}, \ldots$

can be described: $\boxed{a_n = \frac{a_{n-1}}{5}, \quad a_1 = 7}$
$n \geq 2$