13.1 Sequences

Sequence: ordered list (usually a list of numbers)

$\boxed{a_1, a_2, a_3, \ldots}$

Ex: Find first 4 terms of:

a) $\{a_n\} = \left\{ \frac{(-1)^{n+1} e^n}{2} \right\}$

1st term $a_1 = \frac{(-1)^2 e^1}{2} = \frac{e}{2}$ $\qquad a_2 = \frac{(-1)^3 e^2}{2} = -\frac{e^2}{2}$

$a_3 = \frac{(-1)^4 e^3}{2} = \frac{e^3}{2}$ $\qquad a_4 = \frac{(-1)^5 e^4}{2} = -\frac{e^4}{2}$

Sequence: $\frac{e}{2}, -\frac{e^2}{2}, \frac{e^3}{2}, -\frac{e^4}{2}, \ldots$

b) $b_1 = 1 \quad b_2 = 1 \quad \{b_n\} = \{b_{n-1} + b_{n-2}\}$ for $n \geq 3$

1st term $b_1 = 1$ $\qquad b_2 = 1$

$b_3 = b_2 + b_1 = 2$ $\qquad b_4 = b_3 + b_2 = 3$

Sequence: $1, 1, 2, 3, 5, 8, 13, \ldots$

Ex: Write down a_n (n^{th} term)

$-4, 4, -4, 4, -4, \ldots$

$a_n = (-1)^n 4$

Summation Notation or "Sigma Notation"

Ex: Expand

a) $\sum_{k=1}^{4} \frac{1}{k}$ means: the sum of $\frac{1}{k}$ as k ranges from 1 to 4

$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

b) $\sum_{k=2}^{4} \frac{k+1}{4k} = \frac{3}{8} + \frac{4}{12} + \frac{5}{16}$

$(k=2)$ $(k=3)$ $(k=4)$

Ex: Evaluate

a) $\sum_{k=2}^{5} (3k+1) = 7 + 10 + 13 + 16$

$(k=2)$ $(k=3)$ $(k=4)$ $(k=5)$

$= 46$

b) $\sum_{k=1}^{10} (-1)^k = (-1)^1 + (-1)^2 + \cdots + (-1)^9 + (-1)^{10}$

$= \underbrace{-1 + 1} + \cdots + \underbrace{-1 + 1}$

$= 0$

c) $\sum_{k=1}^{n} a = \underbrace{a + a + \ldots + a}_{n \text{ times}} = na$

<u>Ex</u>: Write in summation notation

a) $1 + 8 + 27 + \ldots + 1000$

$= 1^3 + 2^3 + \ldots + 10^3$

$= \sum_{k=1}^{10} k^3$

b) $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{64}$

$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots + \left(\frac{1}{2}\right)^6$

$= \sum_{k=2}^{6} \left(\frac{1}{2}\right)^k$ or $\sum_{k=2}^{6} \frac{1}{2^k}$ or $\sum_{k=1}^{5} \frac{1}{2^{k+1}}$

<u>Properties</u>

$\sum_{k=1}^{n} c a_k = c \sum_{k=1}^{n} a_k$ c: Constant

$\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$

$$\boxed{\begin{array}{l}\text{Formulas} \\ \displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \text{ means } 1+2+\ldots+n = \frac{n(n+1)}{2} \\ \displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \\ \displaystyle\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2 \\ \text{Note: First term must be 1 to apply formula}\end{array}}$$

Ex: Find

a) $$\sum_{k=1}^{20} k = \frac{20(21)}{2} = 210$$

b) $$\sum_{k=5}^{10} k^2 = \sum_{k=1}^{10} k^2 - \sum_{k=1}^{4} k^2$$

$$\boxed{5^2+6^2+\ldots+10^2 = (1^2+2^2+\ldots+10^2) - (1^2+2^2+3^2+4^2)}$$

$$= \left[\frac{n(n+1)(2n+1)}{6}\right]_{n=10} - \left[\frac{n(n+1)(2n+1)}{6}\right]_{n=4}$$

$$= \frac{10(11)(21)}{6} - \frac{4(5)(9)}{6}$$

$$= 355$$

c) $\sum_{k=1}^{7} (2k^2 + k^3)$

$= \sum_{k=1}^{7} 2k^2 + \sum_{k=1}^{7} k^3$ Property #2

$= 2 \sum_{k=1}^{7} k^2 + \sum_{k=1}^{7} k^3$ Property #1

$= 2 \left[\frac{n(n+1)(2n+1)}{6} \right]_{n=7} + \left[\frac{n(n+1)}{2} \right]^2_{n=7}$

$= 2 \cdot \frac{7(8)(15)}{6} + \left[\frac{7(8)}{2} \right]^2$

$= 1064$

<u>Factorials</u>

$0! = 1$ $1! = 1$ $n! = n(n-1) \cdots 1$

$2! = 2 \cdot 1 = 2$
$3! = 3 \cdot 2 \cdot 1 = 6$
$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40{,}320$